Accuracy Issues in Space Weather MHD Models

*** How to estimate computational errors?**

*** How to improve accuracy/precision?**

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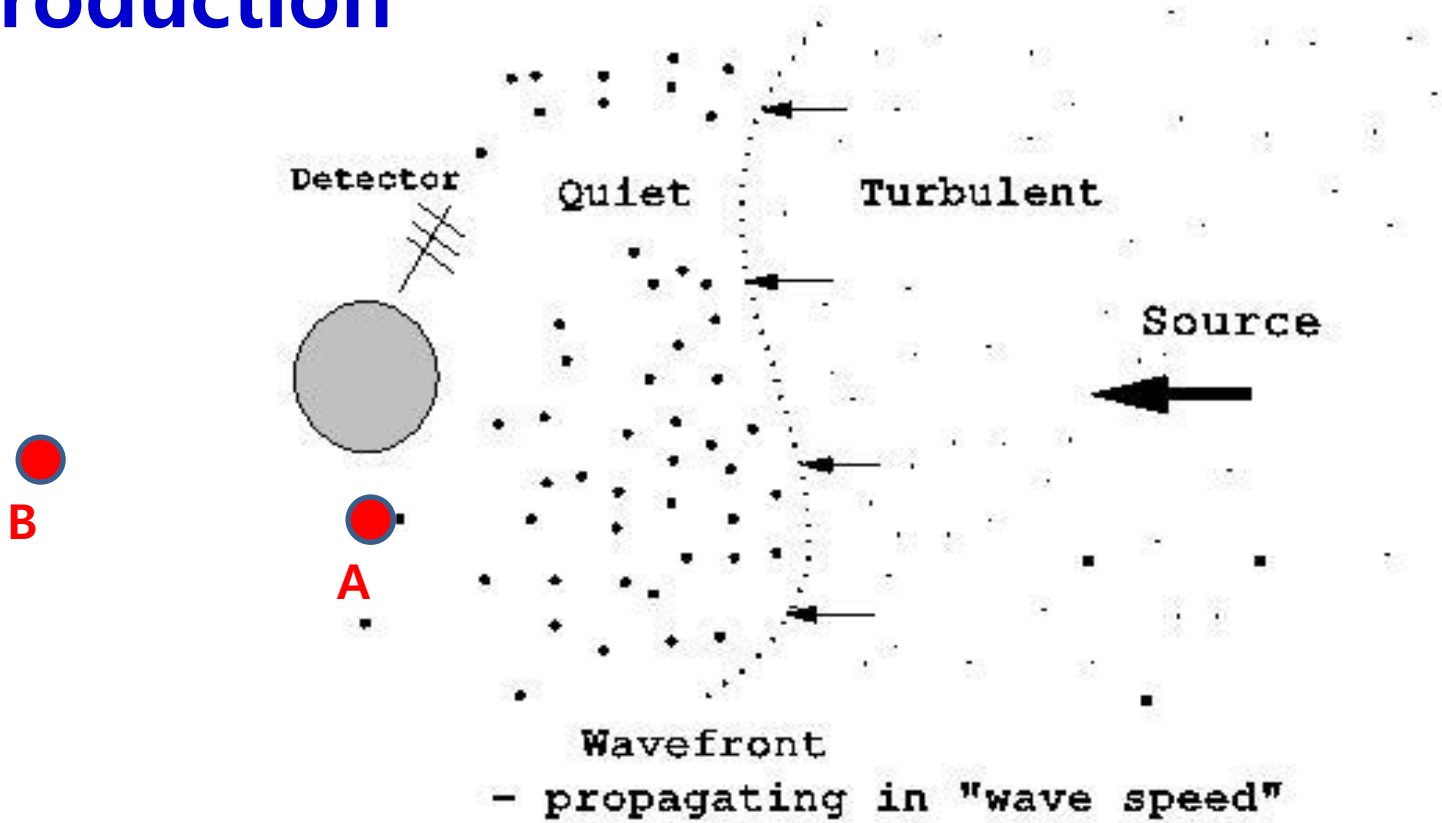
- **Introduction:**
  * **Effects of nonlinearity in a time-dependent system**

- **Theory:**
  * **What kinds of *exact* MHD solutions available?**

- **Numerical tests:**
  * **Theory *vs.* Numerical solutions**
  * **Effects of grid resolution**

- **Conclusion**

Introduction



- MHD numerical models are often studied in the interplanetary space. (*e.g., ENLIL*)
- As realistic variations in the solar wind become often nonlinear, it is important to investigate time-dependent behaviors of the solar wind fluctuations.

Introduction

$$f \gg \delta f$$

- If the disturbance is <u>linear</u> in a <u>uniform</u> space, f_A = f_B:

A B

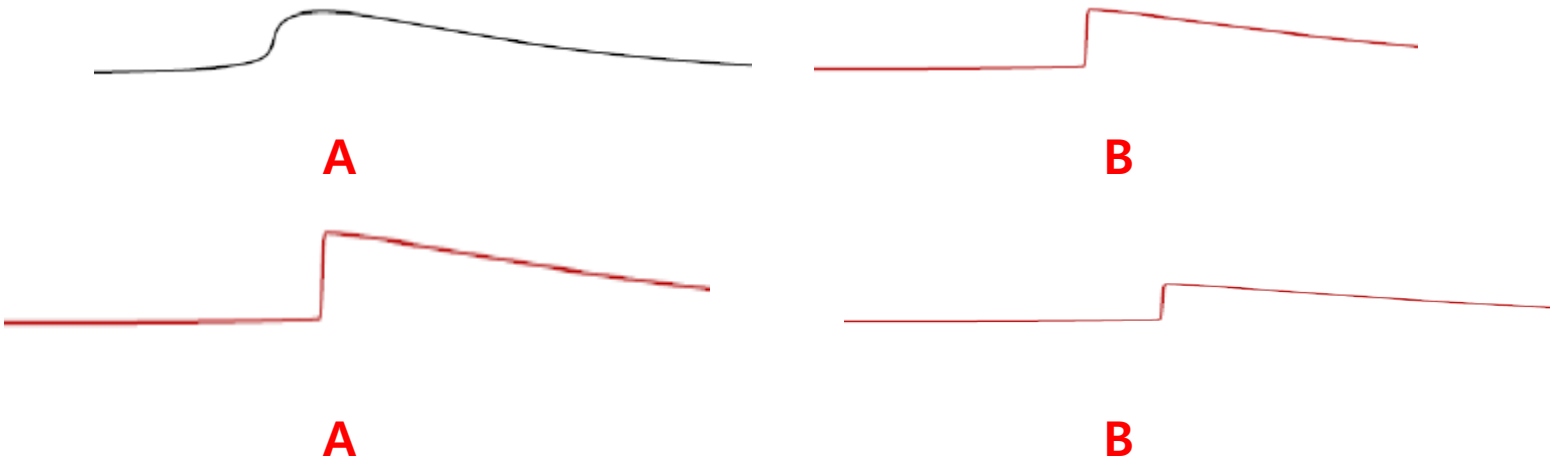
- If the disturbance is <u>linear</u> in a <u>nonuniform</u> space, only the effects of refraction/reflection are to change f_A & f_B:

A B

Introduction

$$f \sim \delta f$$

- If the disturbance is <u>nonlinear</u>, f_A and f_B become differentiated even in a <u>uniform</u> space:



- The disturbance in the SW rest frame can evolve in a time-dependent manner, which is <u>far from the steady-state</u>.

ex) *Rankine-Hugoniot relations*
Numerical test with stationary shock structure

- **For instance, the eq. of motion of MHD:**

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} v = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} (\vec{\nabla} \times \vec{B}) \times \vec{B}$$

$$\frac{\partial \vec{v}}{\partial t} >> \vec{v} \cdot \vec{\nabla} v$$ **: Linear MHD waves** $f >> \delta f$

$$\frac{\partial \vec{v}}{\partial t} << \vec{v} \cdot \vec{\nabla} v$$ **: Steady-state** ***cf) R-H relations***

$$\frac{\partial \vec{v}}{\partial t} \sim \vec{v} \cdot \vec{\nabla} v$$ **: Nonlinear MHD waves** $f \sim \delta f$

Theory

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} v + \frac{1}{\rho} \nabla p - \frac{1}{\rho} (\vec{\nabla} \times \vec{B}) \times \vec{B} = 0$$

$$\frac{\partial p}{\partial t} + v \cdot \nabla p + \gamma p \nabla \cdot v = 0$$

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{B}) = 0$$

- **The assumption of simple waves (similarity flow) is often used to obtain a solution.**

- **Exact solution for the nonlinear MHD wave is available if it is a one-dimensional uniform system.**

Theory

Sources → **Simple waves** → **Shock waves**

$(t > 0)$ $(t = \tau_S)$

* **Simple waves** $(0 < t \le \tau_S)$

$$\rho, \vec{v}, \vec{B}, p, \ldots\ldots = f_i(x - vt)$$

or

$$x - vt = g_i(\rho, \vec{v}, \vec{B}, p, \ldots\ldots)$$

$$\text{where} \quad v = v(\rho, \vec{v}, \vec{B}, p, \ldots\ldots)$$

* Simple waves

if $\vec{k} \,//\, \hat{x}$

- **Alfven waves:**

$$x - \left[v_x + \frac{B_x}{\sqrt{\mu_o \rho}} \right] t = \alpha(v_y, v_z)$$

where $\rho, \quad v_x, \quad |B|, s = \text{const}$

- **Compressional waves:**

$$x - [v_x + v_\pm] t = \beta(\rho)$$

where $\dfrac{B_y}{B_z} = \dfrac{dv_y}{dv_z} = \text{const}$

- **Entropy waves:**

$$s = \gamma(x - v_x t)$$

where $\rho, \vec{v}, \vec{B} = \text{const}$

α, β, γ **are determined by IC and BC.**

Theoretical solutions

- **MHD equations**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0,$$
$$\frac{\partial B}{\partial t} + \frac{\partial(B v)}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p^*}{\partial x} = 0,$$
$$p^* = p + B^2/2\mu_0$$

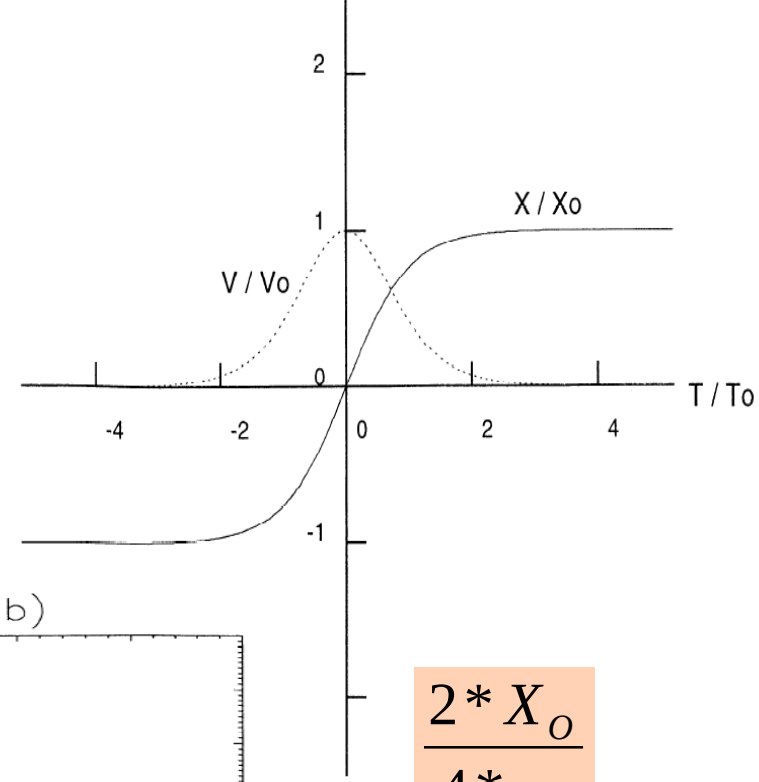
- **Exact solution for the MHD wave** **[Lee & Kim, JGR, 2000]**

$$x = X(\tau) + (t - \tau)\left\{\dot{X}(\tau) \pm C_T\left[\dot{X}(\tau)\right]\right\}$$
$$v = \dot{X}(\tau)$$
$$v = \pm 2V_{A_0}\sqrt{\frac{\rho}{\rho_0}}\left[1 + \frac{C_{S_0}^2}{V_{A_0}^2}\left(\frac{\rho}{\rho_0}\right)^{-\frac{1}{3}}\right]^{\frac{3}{2}} + v_0$$
$$\frac{C_T}{V_{A_0}} = \frac{v - v_0}{2V_{A_0}} - \frac{C_{S_0}^2}{2^{\frac{1}{3}}V_{A_0}^2}\left(\frac{v - v_0}{V_{A_0}}\right)^{\frac{1}{3}}$$

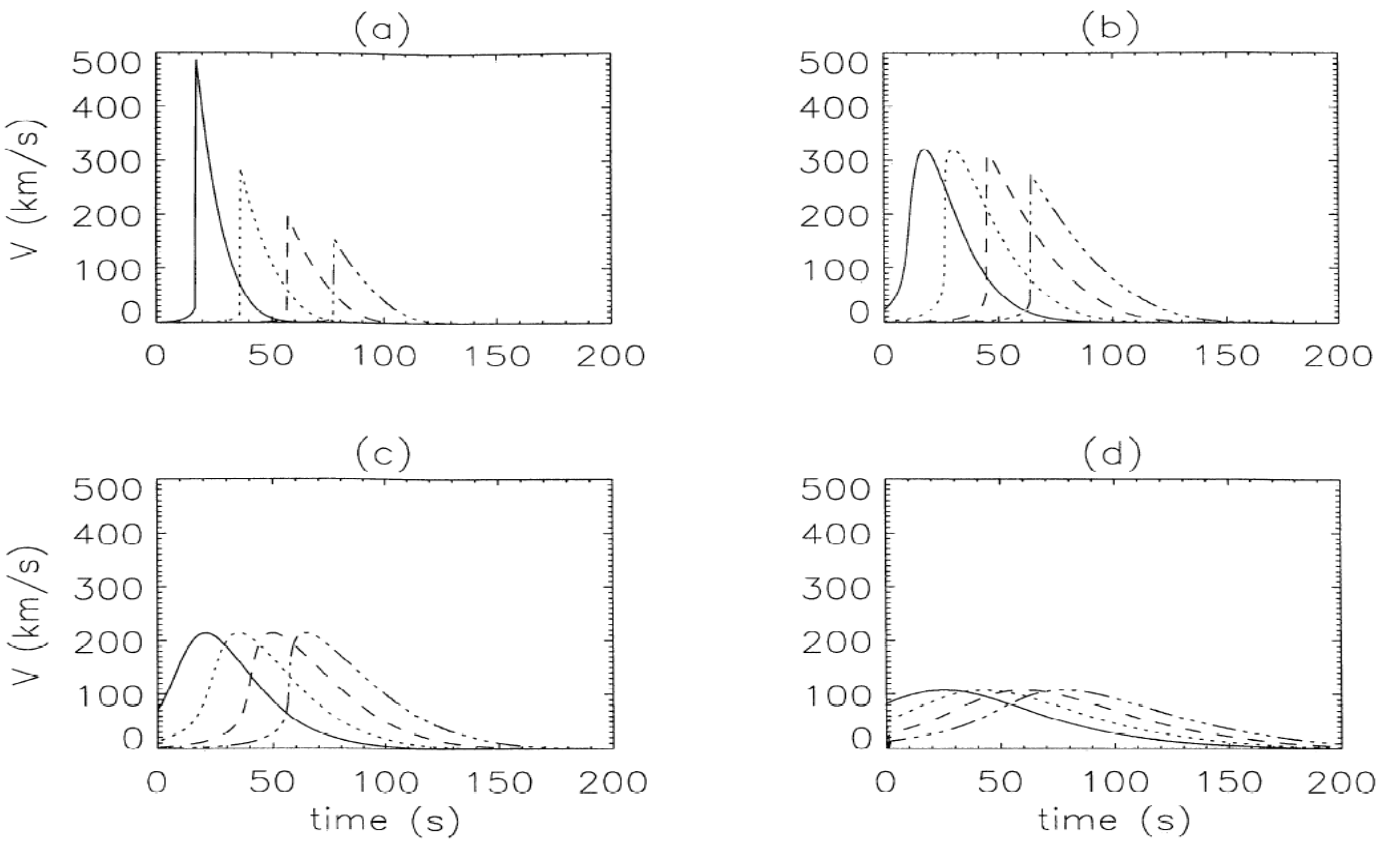
Ex) piston-like motion

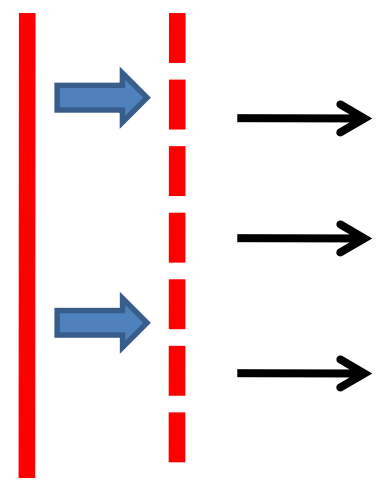
$$X(t) = X_0 \tanh\left(\frac{t}{\tau_0}\right) \qquad v = \frac{X_0}{\tau_0}\sec h^2\left(\frac{t}{\tau_0}\right)$$

$$\frac{x}{X_0} = \tanh\frac{\tau}{\tau_0} + \left(\frac{t}{\tau_0} - \frac{\tau}{\tau_0}\right)\left(\sec h^2\frac{\tau}{\tau_0} \pm \frac{C_T}{X_0/\tau_0}\right)$$

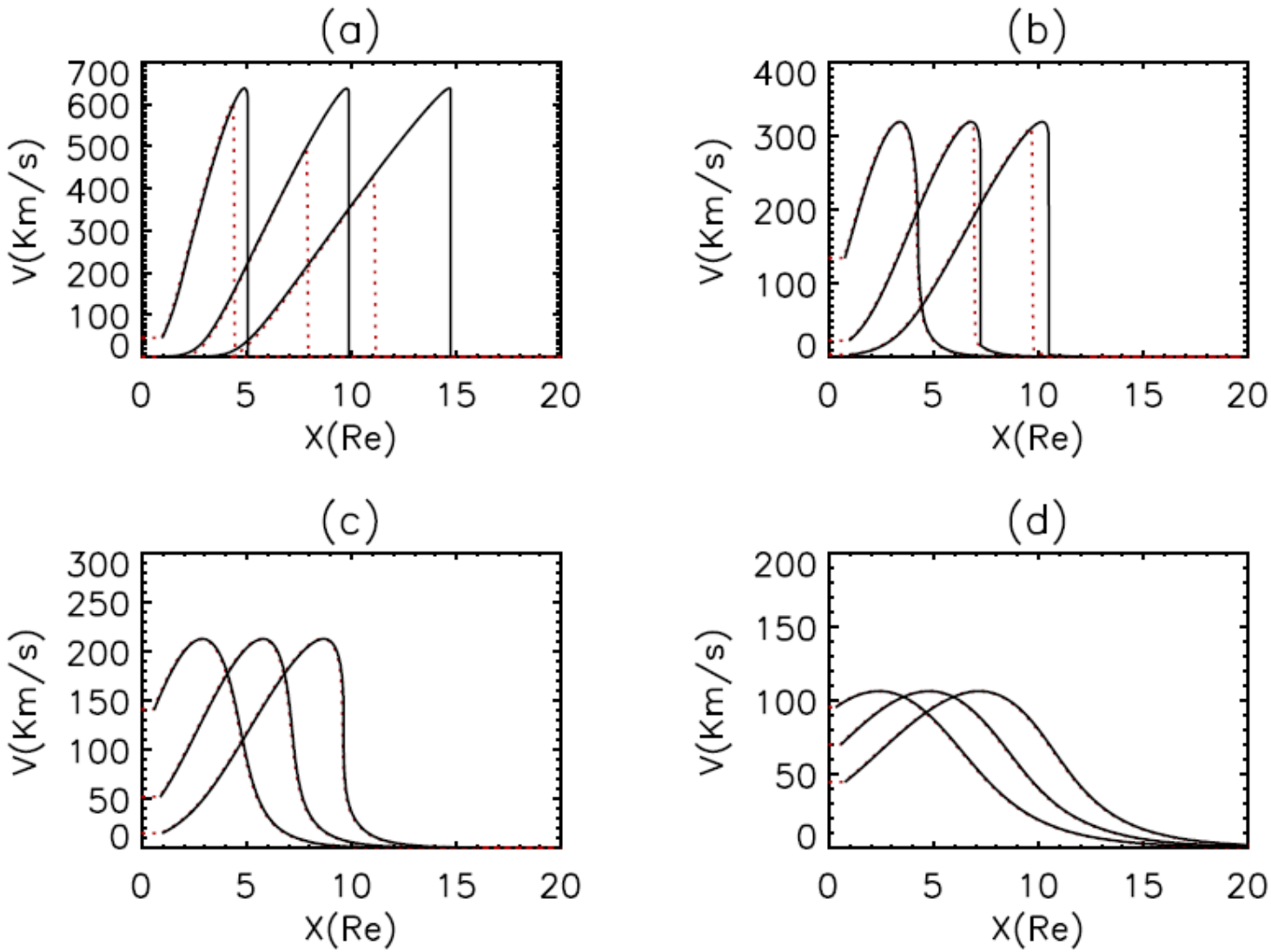


$$\frac{2 * X_O}{4 * \tau_O}$$



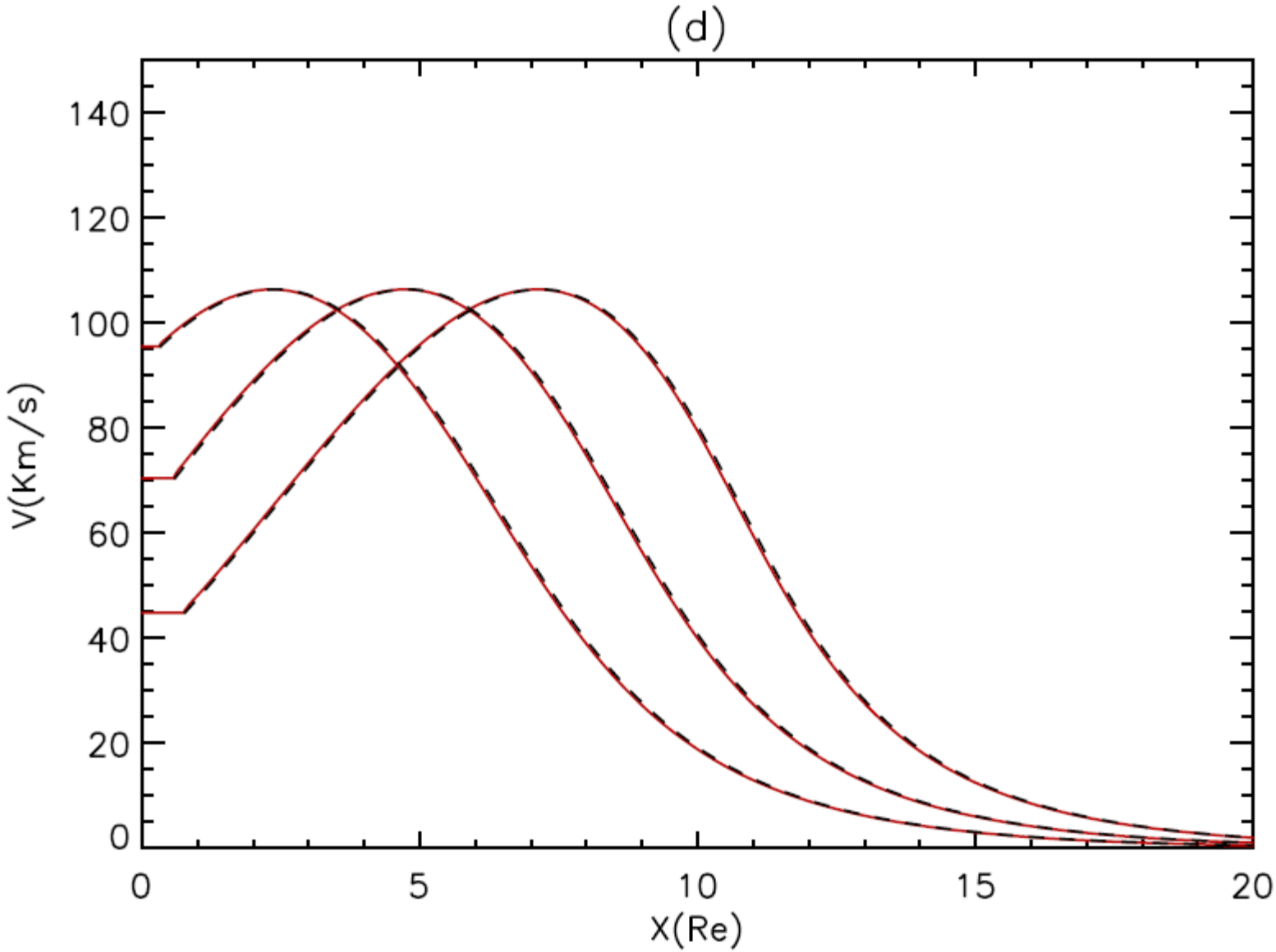


Numerical test *vs.* Exact solutions



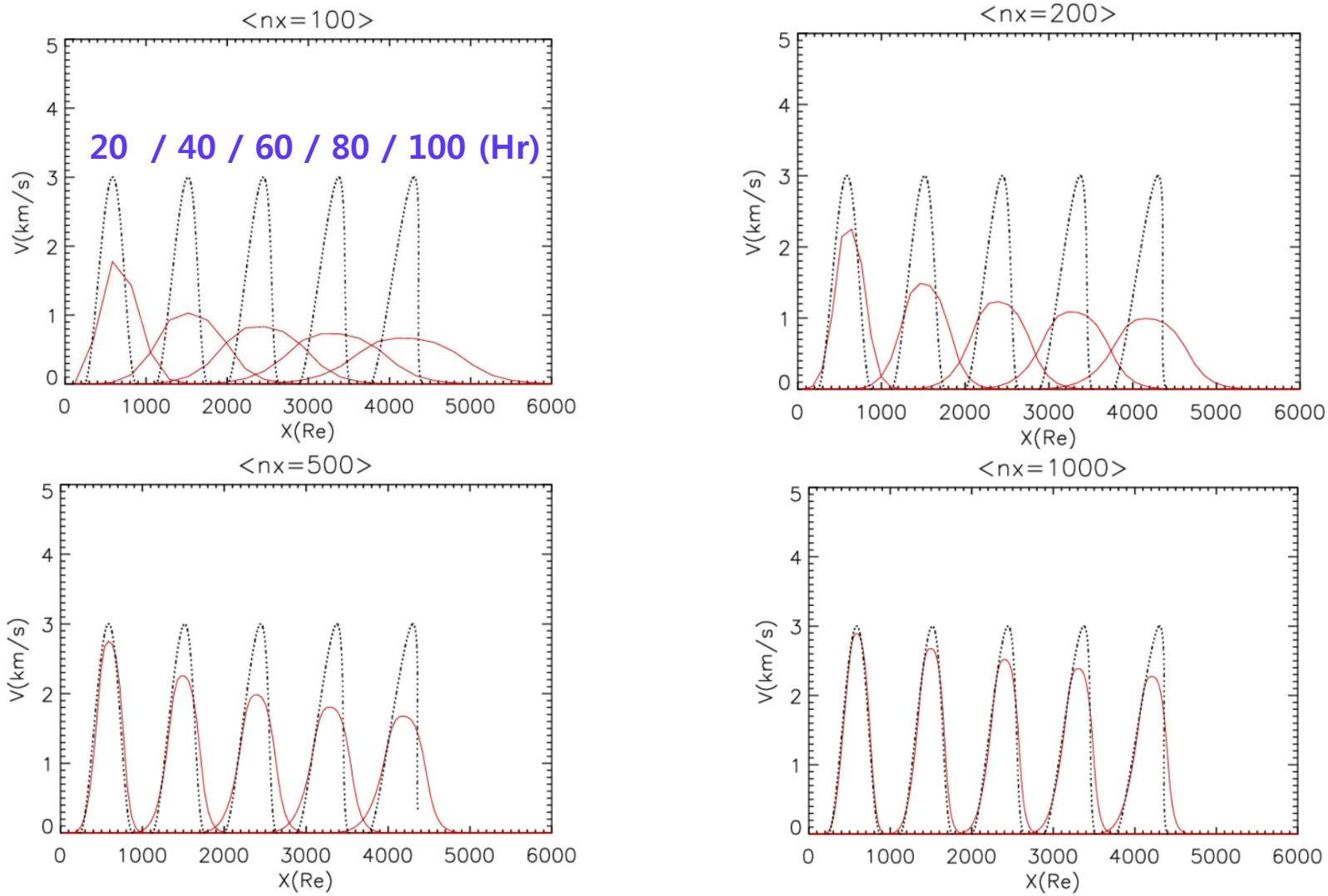
Numerical solutions are corresponding to the exact solutions before the shock formation!

Numerical test *vs.* Exact solutions

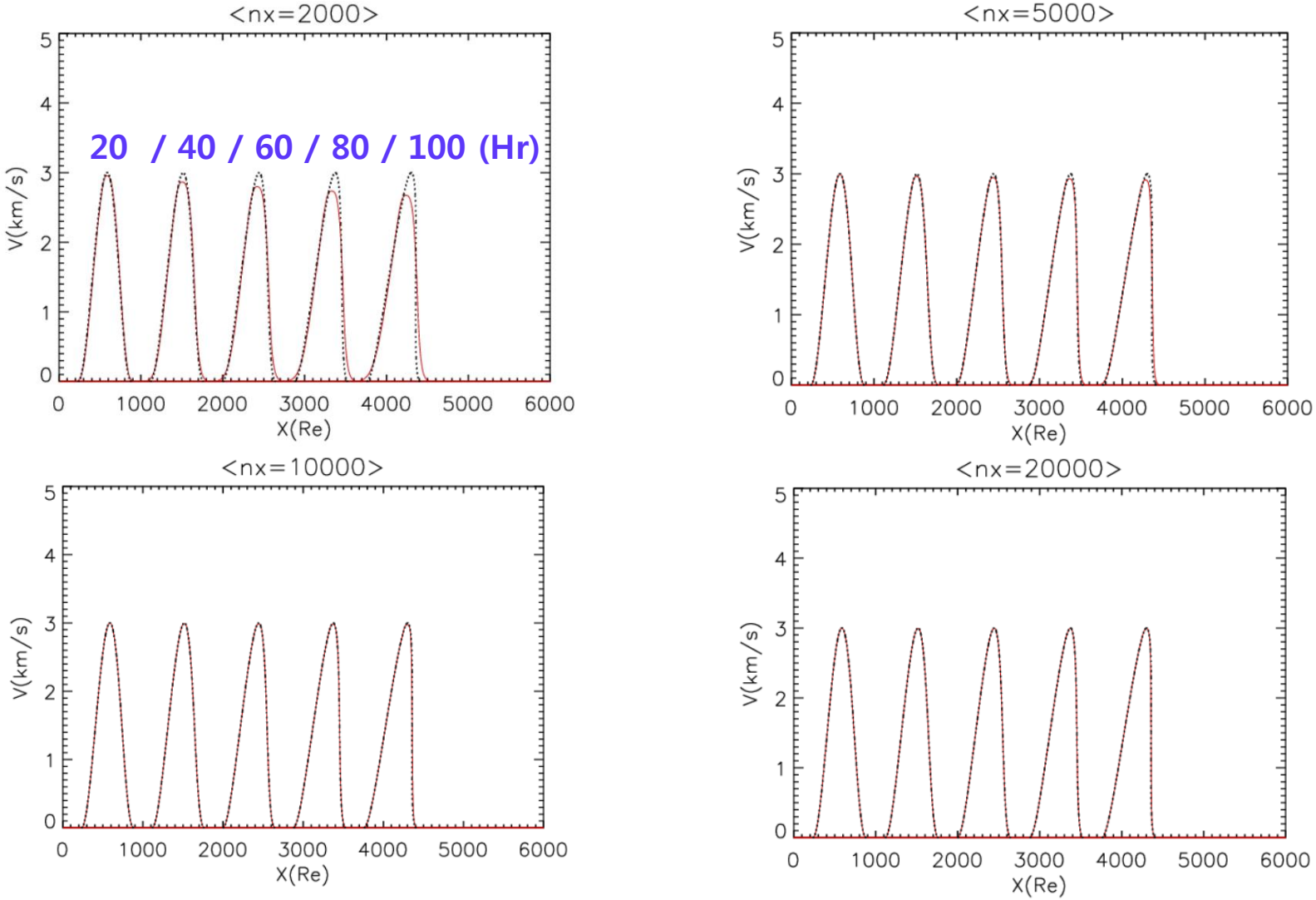


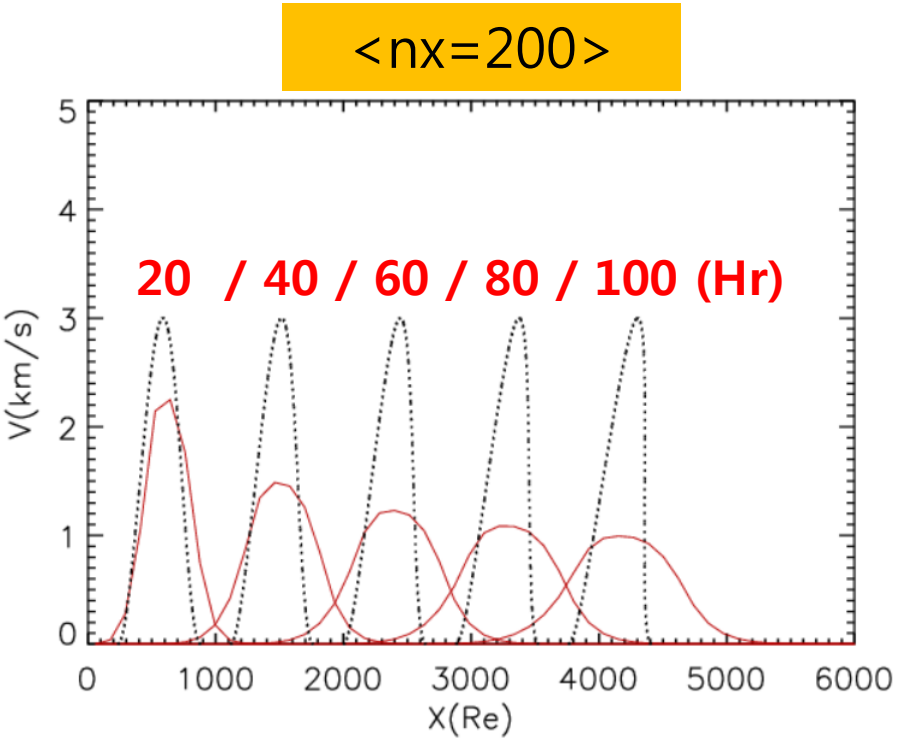
Numerical solutions = Analytic solutions

Numerical test *vs.* Exact solutions



Numerical test *vs.* Exact solutions

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20 / 40 / 60 / 80 / 100 (Hr)
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3
4
5
0
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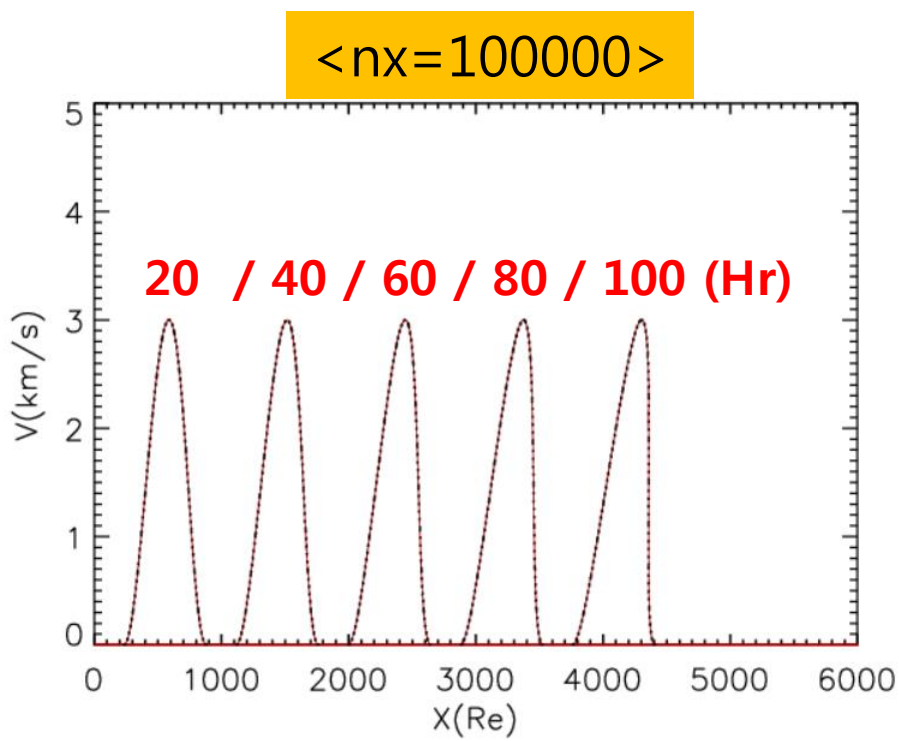
3000
4000
5000
6000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X(Re)
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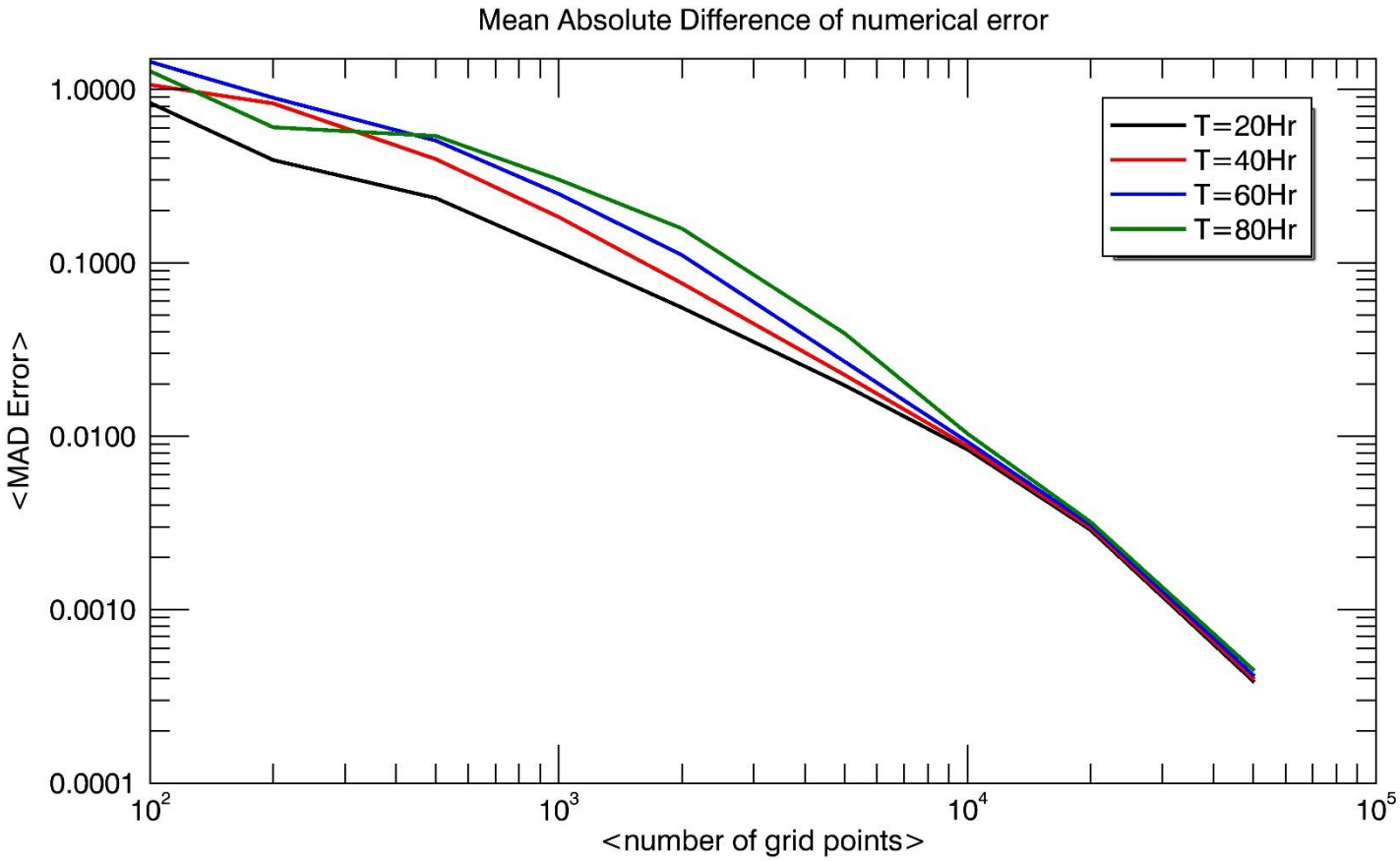


$$\langle | \delta F | \rangle = \frac{\sum_i | f(x_i) - \mathrm{F}(x_i) |}{N}$$

Introduction: *Numerical models*

[Toth, 2000]

TABLE I

Basic Properties of Schemes

Name	Base scheme	$\nabla \cdot \boldsymbol{B} = 0$ constraint	Cost[a]	Description	Reference
8-wave	Any	Truncation error	7%	Section 3	[31, 15]
Field-CT	Any	Conserves (15) and (27)	4%	Subsections 4.2 and 4.4	DW [11, 12]
Flux-CT	Any	Conserves (15) and (27)	5%	Subsections 4.3 and 4.4	BS [3]
Tr-flux-CT	One step TVD	Conserves (15) and (27)	6%	Subsections 4.3 and 4.4	RMJA [35]
Field-CD	Any	Conserves (26)	2%	Subsection 4.5	This paper
Flux-CD	Any	Conserves (26)	4%	Subsection 4.5	This paper
Projection	Any	Enforces (26) to be $<\epsilon$	≈20%	Section 5	[7]

TABLE II

Convergence of Average Errors for Alfvén Waves

Travelling waves	$\bar{\delta}_8$	$\bar{\delta}_{16}$	$\bar{\delta}_{32}$	$\bar{\delta}_{64}$	Standing waves	$\bar{\delta}_8$	$\bar{\delta}_{16}$	$\bar{\delta}_{32}$	$\bar{\delta}_{64}$
Projection	0.716	0.135	0.032	0.012	Projection	0.299	0.079	0.031	0.012
Base scheme	0.711	0.133	0.033	0.012	Base scheme	0.321	0.112	0.034	0.013
8-wave	0.713	0.134	0.033	0.012	Flux-CD/CT	0.315	0.122	0.037	0.013
Tr-flux-CT	0.852	0.193	0.044	0.014	Field-CT	0.268	0.102	0.044	0.021
Field-CD	1.016	0.336	0.087	0.025	Field-CD	0.294	0.106	0.044	0.021
Flux-CD/CT	0.971	0.393	0.105	0.029	8-wave	0.351	0.127	0.050	0.024
Field-CT	0.927	0.566	0.163	0.044	Tr-flux-CT	0.549	0.287	0.111	0.046

TABLE VII

Convergence of Averaged Errors in the Orszag–Tang Test

$t = 1$	$\bar{\delta}_{50}$	$\bar{\delta}_{100}$	$\bar{\delta}_{200}$	$t = 3.14$	$\bar{\delta}_{50}$	$\bar{\delta}_{100}$	$\bar{\delta}_{200}$
Field-CD	0.0250	0.0085	0.0026	Field-CD	0.1150	0.0617	0.0300
Field-CT	0.0284	0.0096	0.0029	Projection	0.1280	0.0709	0.0340
Projection	0.0287	0.0109	0.0035	Field-CT	0.1393	0.0720	0.0355
Flux-CT	0.0321	0.0114	0.0035	Flux-CT	0.1352	0.0737	0.0358
8-wave	0.0305	0.0119	0.0043	Flux-CD	0.1380	0.0775	0.0373
Flux-CD	0.0347	0.0122	0.0037	Tr-flux-CT	0.1527	0.0837	0.0439
Base scheme	0.0324	0.0128	0.0044	8-wave	0.1420	0.0867	0.0478
Tr-flux-CT	0.0349	0.0134	0.0045	Base scheme	0.1784	0.1097	0.0705[a]

[a] Obtained with the minmod limiter.

TABLE IX

Numerical Errors Relative to the Most Accurate Scheme for Each Test

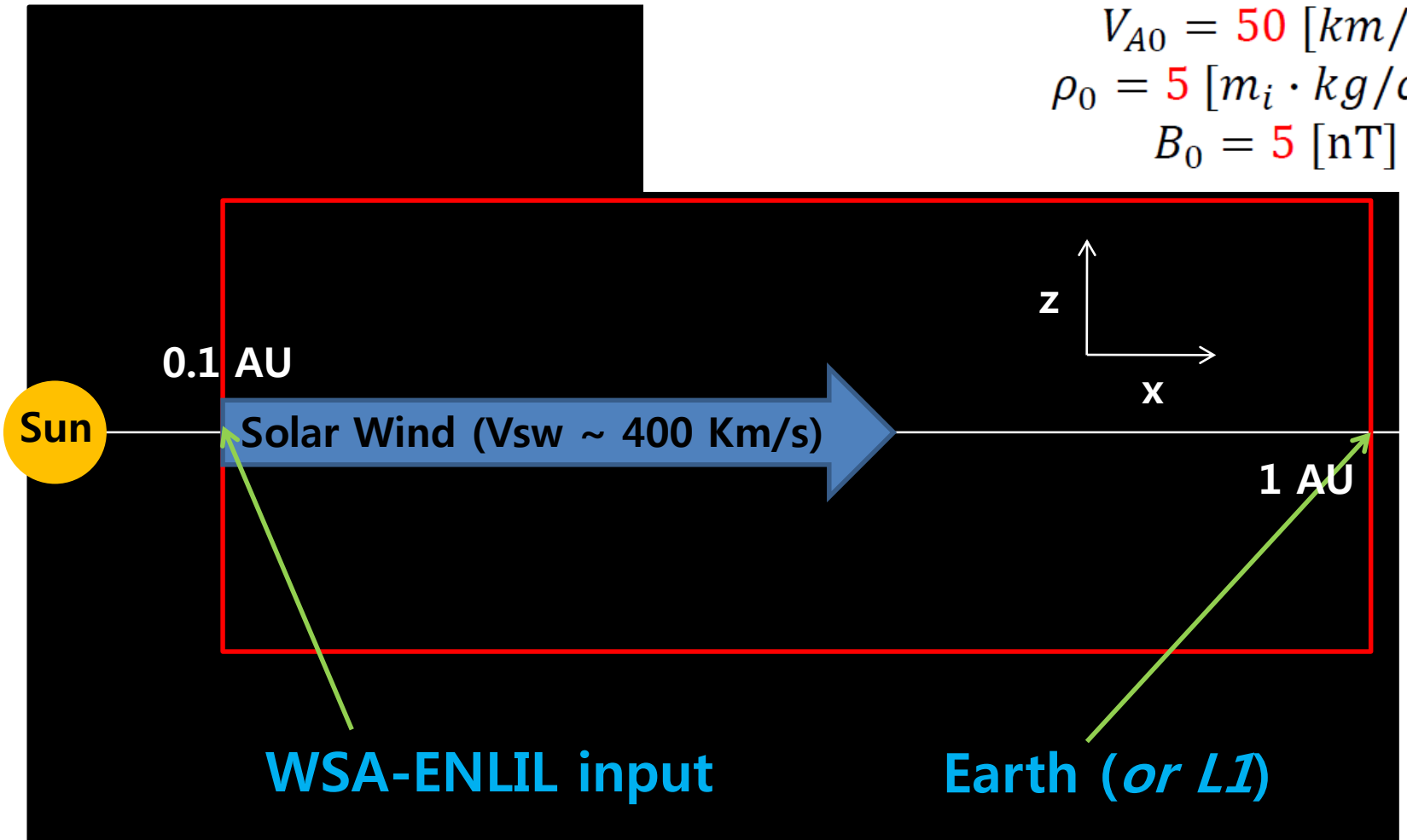
	Project.	Field-CD	Flux-CT	Flux-CD	8-wave	Tr-flux-CT	Field-CT	Base
Rotated 1D tests								
Alfvén travelling	1.000	2.177	2.491	2.491	1.005	1.290	3.562	1.002
Alfvén standing	1.000	1.374	1.219	1.219	1.599	3.221	1.339	1.168
2D shock $\alpha = 63°$	1.022	1.005	1.000	1.007	1.268	1.014	1.269	1.006
2D shock $\alpha = 45°$	1.000	1.031	1.047	1.047	1.801	1.080	1.298	1.782
2.5D shock tube	1.000	1.137	1.234	1.234	1.023	1.132	1.392	1.333
True 2D tests								
Orszag $t = 1$	1.259	1.000	1.324	1.415	1.425	1.568	1.127	1.498
Orszag $t = 3.14$	1.132	1.000	1.188	1.233	1.411	1.383	1.187	1.893[a]
Cloud-shock	1.007	1.069	1.000	1.036	1.013	—	1.072[b]	1.348[a]
Rotor $p = 1$	1.000	1.220	1.052	1.216	1.023	—	1.530	1.094
Rotor $p = 0.5$	1.000	1.058	1.098	1.116	1.050	1.230[a]	1.289	1.071
Correction factor	≈1.06	1.006	1.018	1.013	1.023	1.022	1.014	1.000

Simulation Box (Lx=1AU)

- Simulation parameters

cf) N = 100, 200, 250, 500,
1000, 2500, 5000, 10000

$$V_B = 400\ [km/s]$$
$$C_{s0} = 60\ [km/s]$$
$$V_{A0} = 50\ [km/s]$$
$$\rho_0 = 5\ [m_i \cdot kg/cm^3]$$
$$B_0 = 5\ [\mathrm{nT}]$$



- Source motion

$$v = v_0 \sin^2(w_0 t)$$

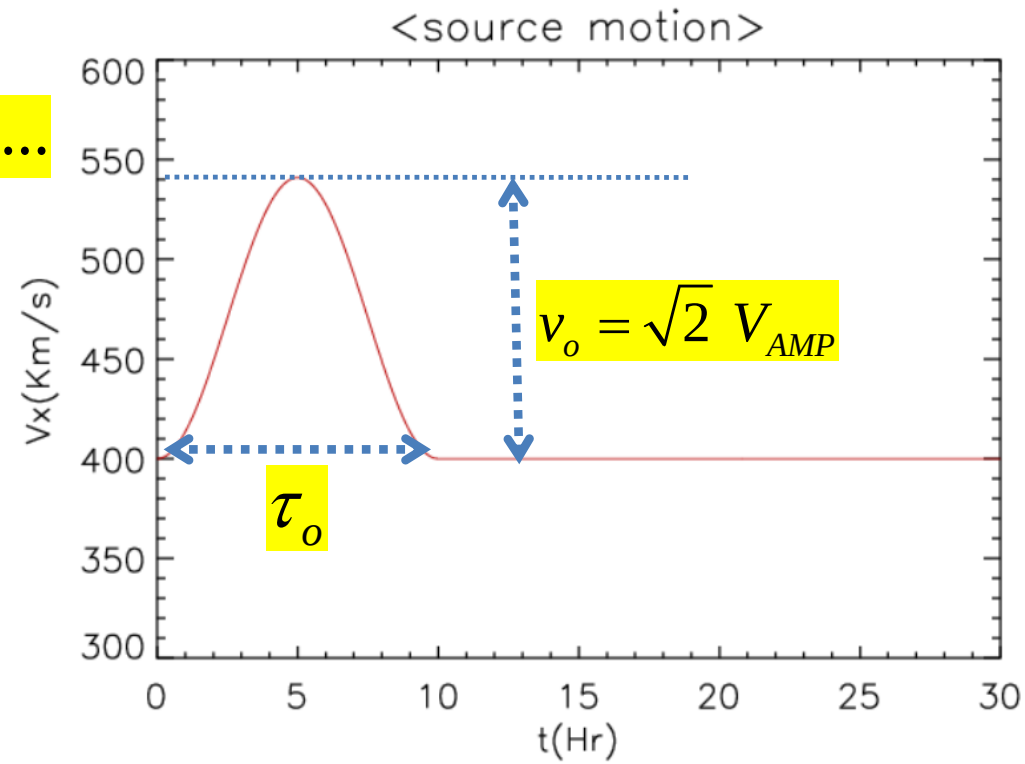
Where, $w_0 = \frac{\pi}{\tau_0}, \quad t = 0 \sim \tau_0$

$$V_{Amp} \equiv \sqrt{< v^2 >} = v_0/\sqrt{2}$$

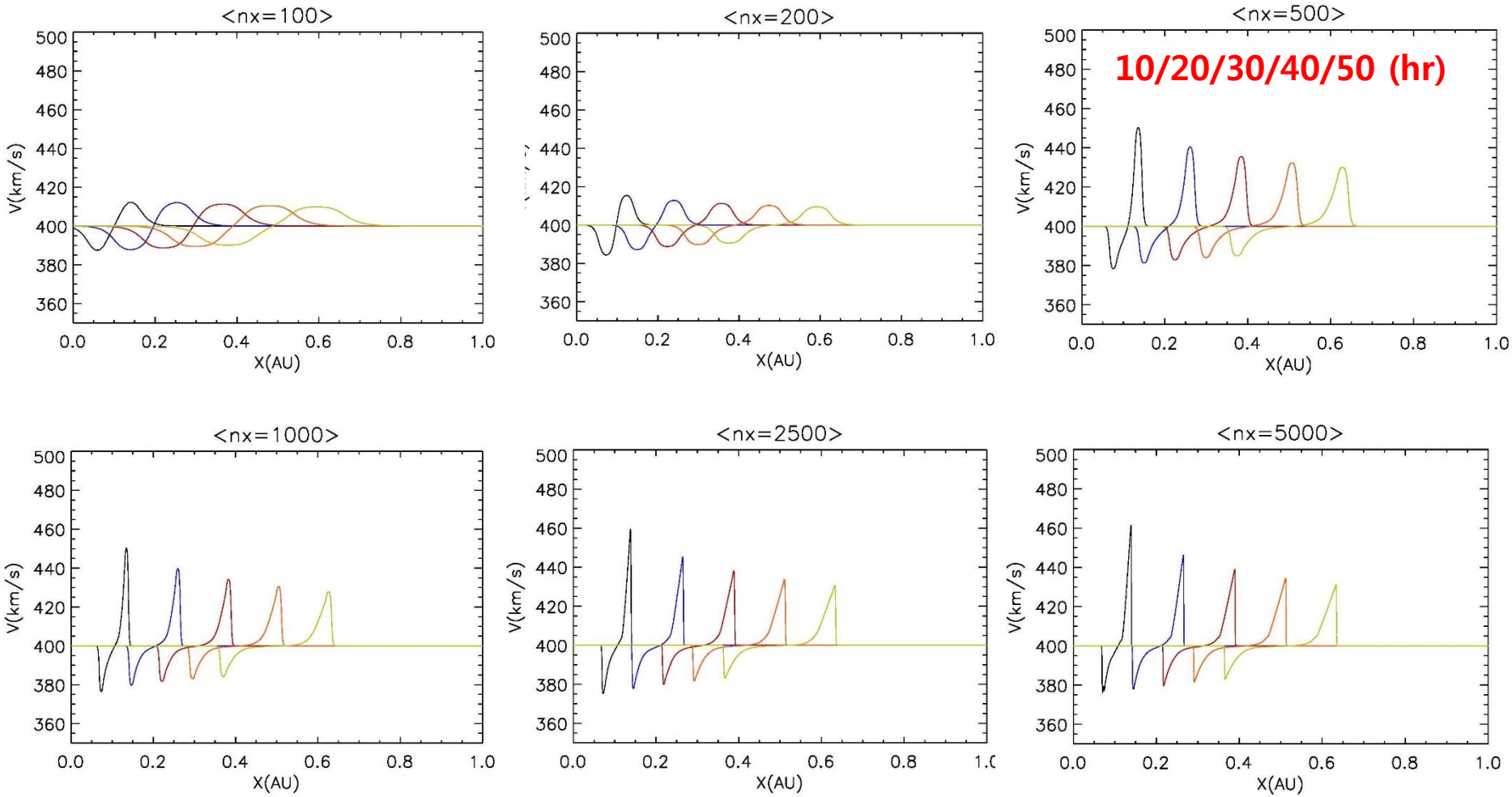
$\tau_o = 10\,\text{min},\ 20\text{min},\ 1\,\text{hr},\ 3\,\text{hr},\ \ldots$

$V_{AMP} = 300, 500, 700, 900,$
$1200, 1500\ \text{km/s}$

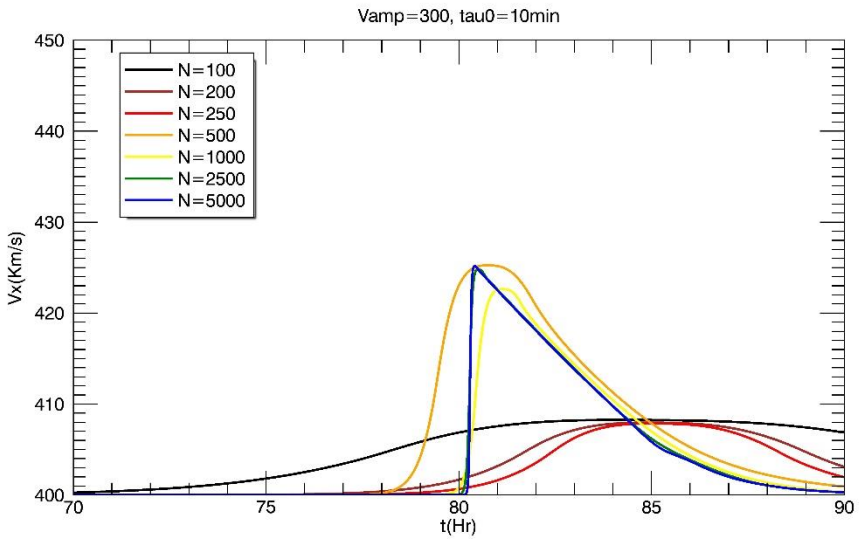
cf) N = 100, 200, 250, 500,
1000, 2500, 5000, 10000



$V_{AMP} = 300$ km/s $\tau_o = 10$ min



$V_{AMP} = 300\ Km/s$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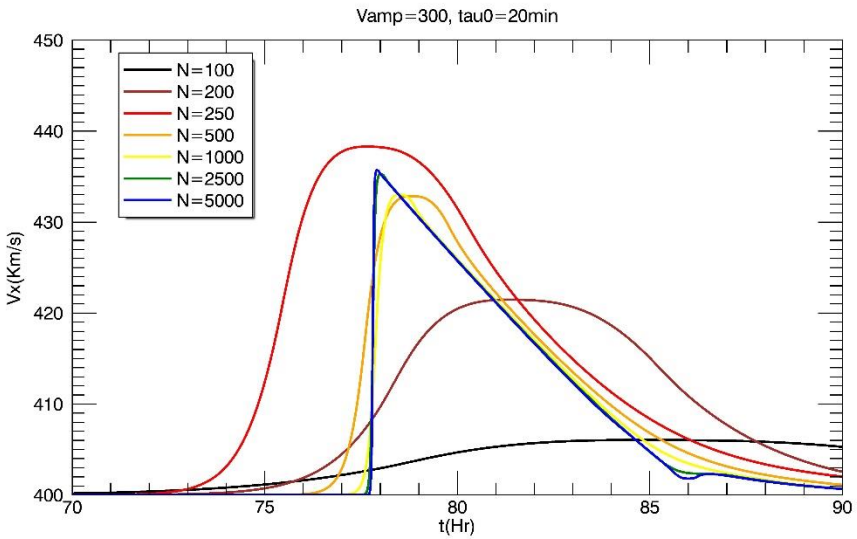
Vx(Km/s)
t(Hr)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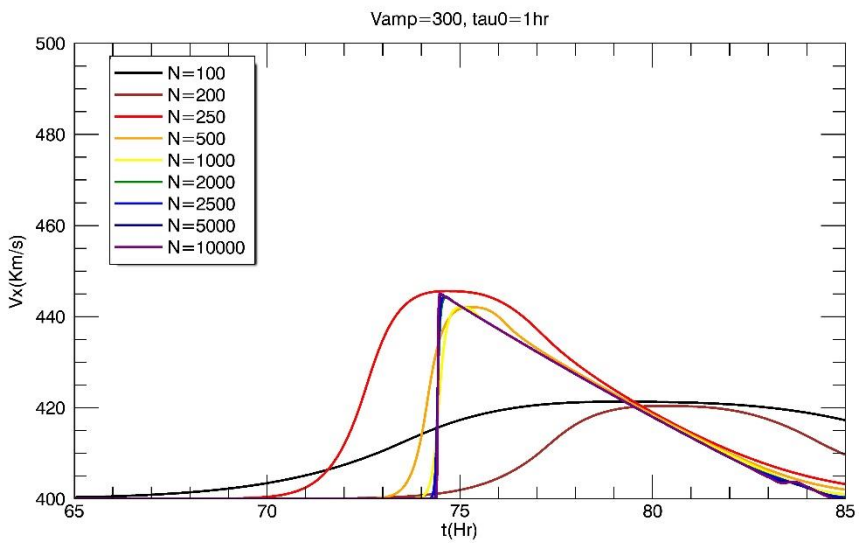
Vx(Km/s)
t(Hr)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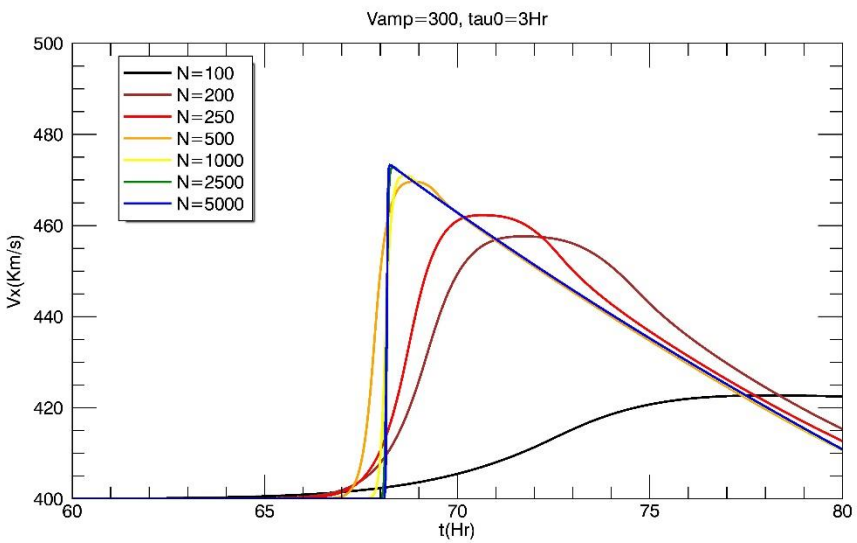
N=10000
Vx(Km/s)
t(Hr)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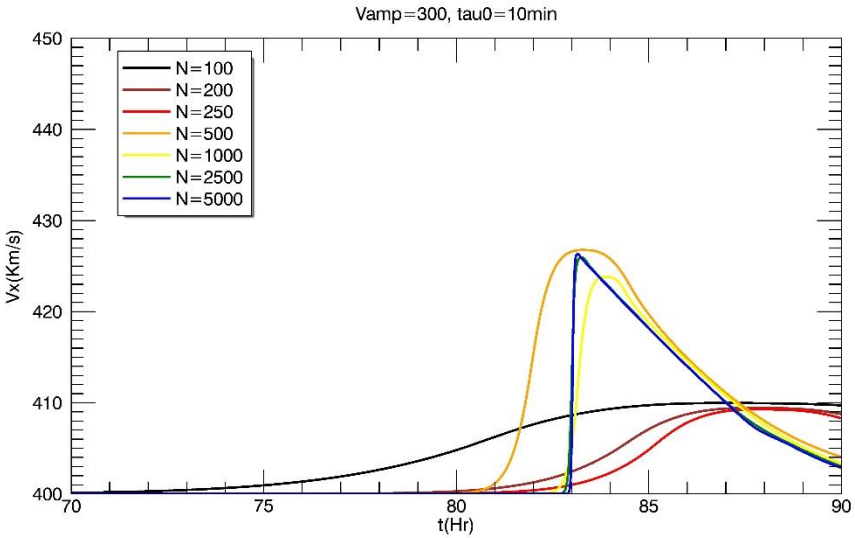
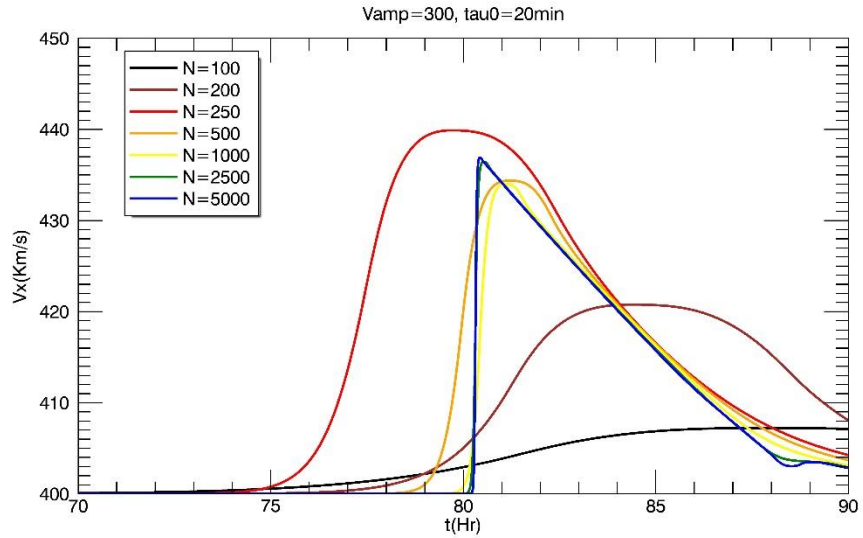
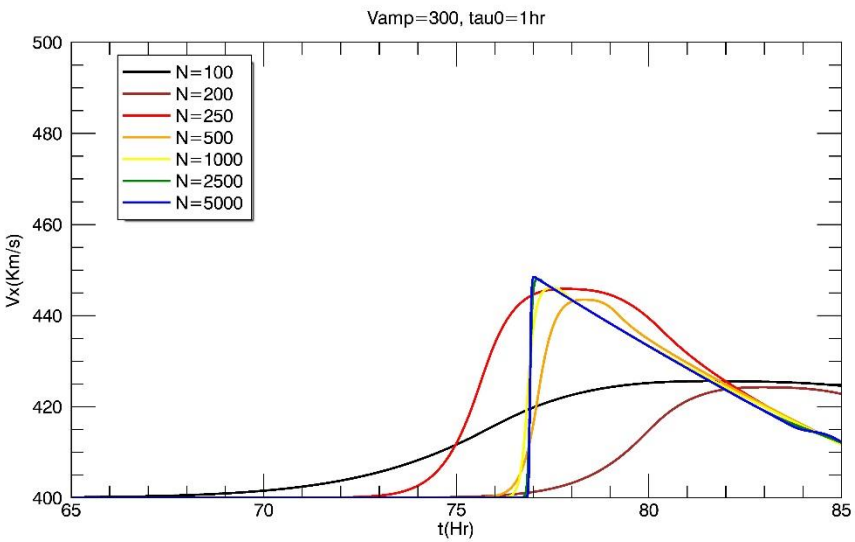
N=5000
Vx(Km/s)
t(Hr)

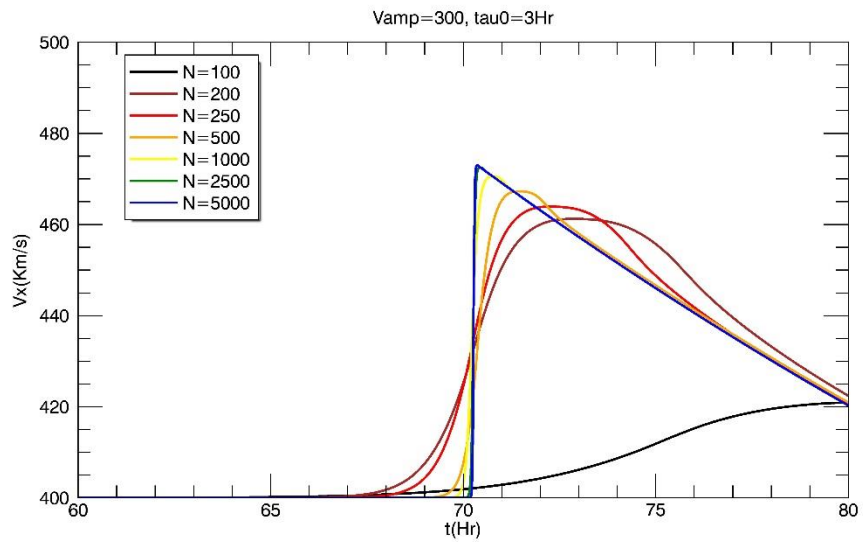
B=Bx

$$V_{AMP} = 300\ Km/s$$

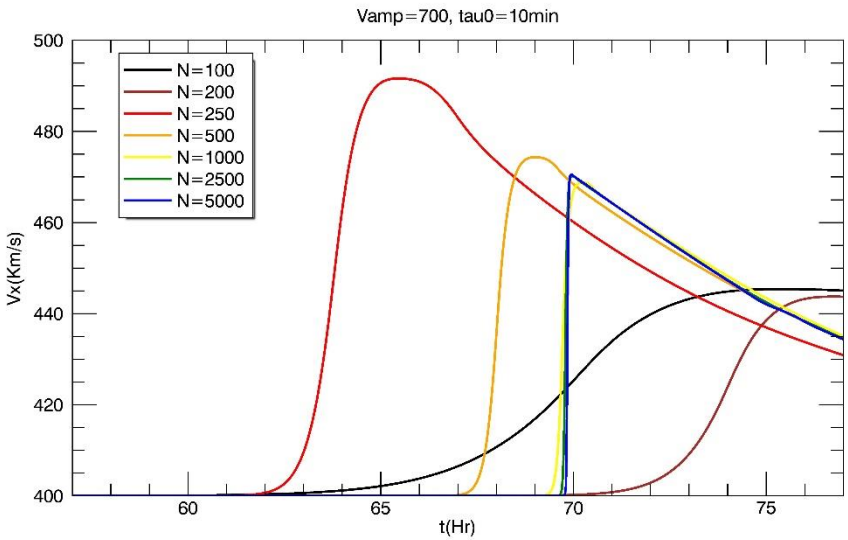








$V_{AMP} = 700\ Km/s$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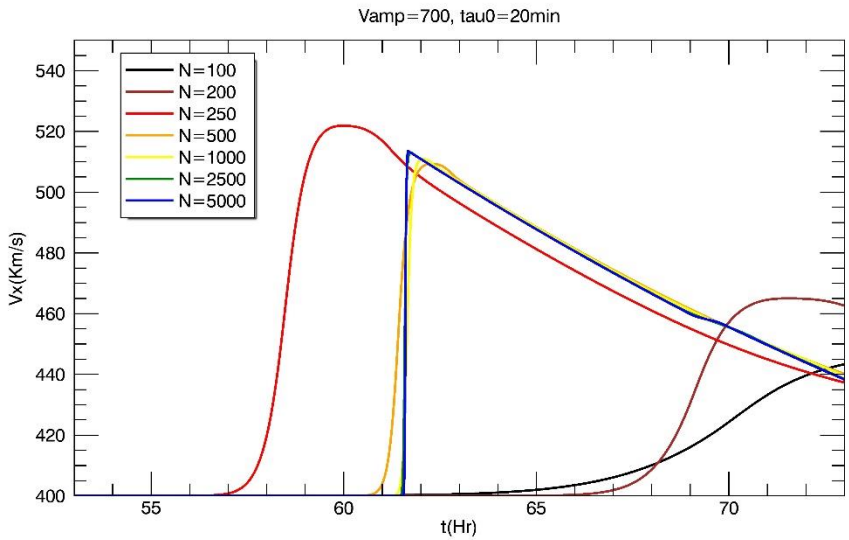
Vx(Km/s)
t(Hr)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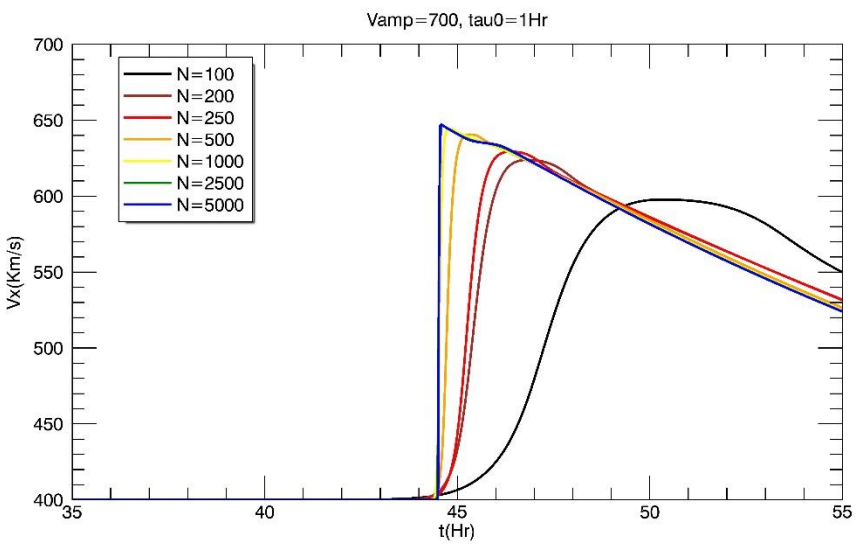
Vx(Km/s)
t(Hr)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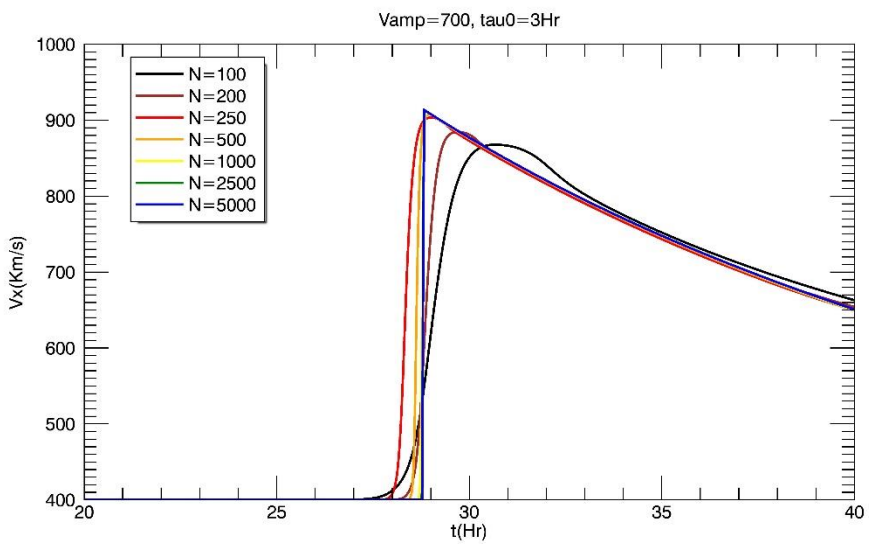
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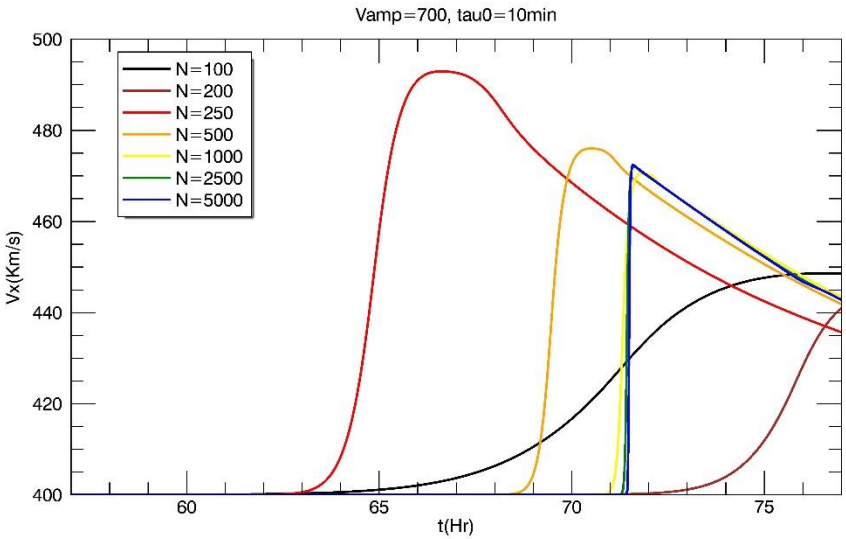
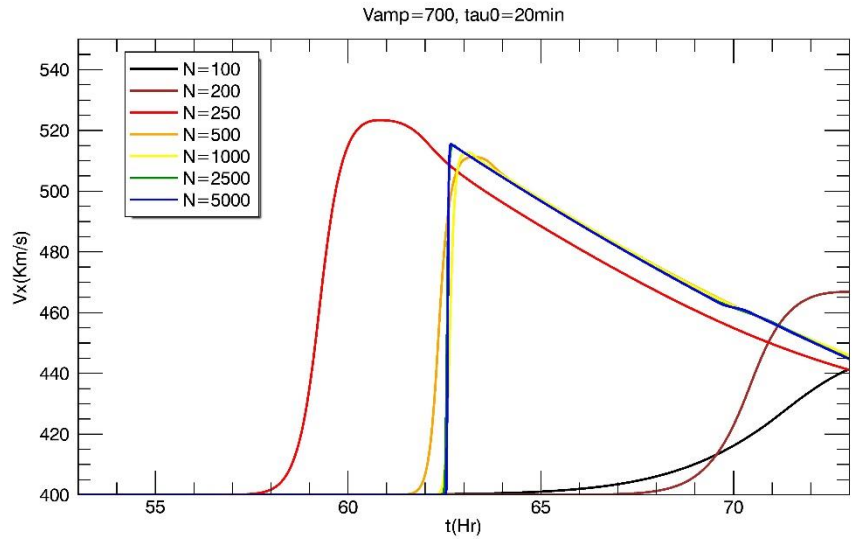
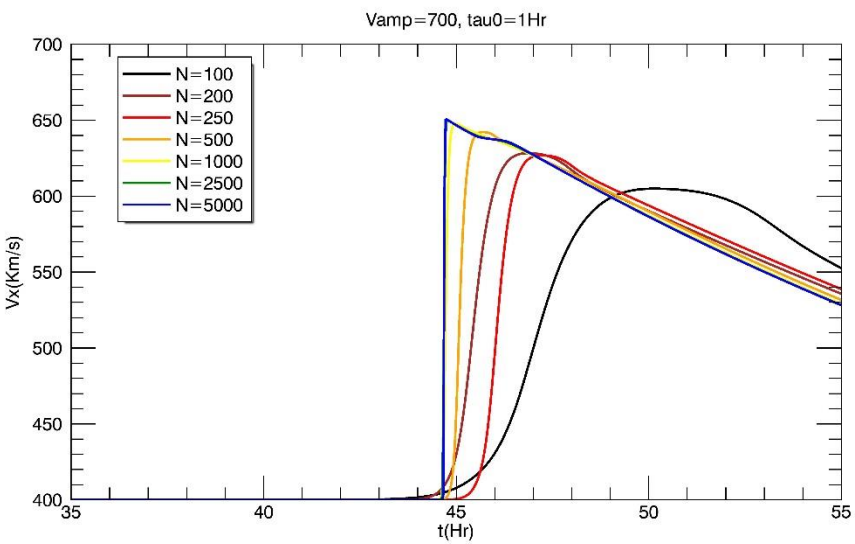
N=2500
N=5000
Vx(Km/s)
t(Hr)

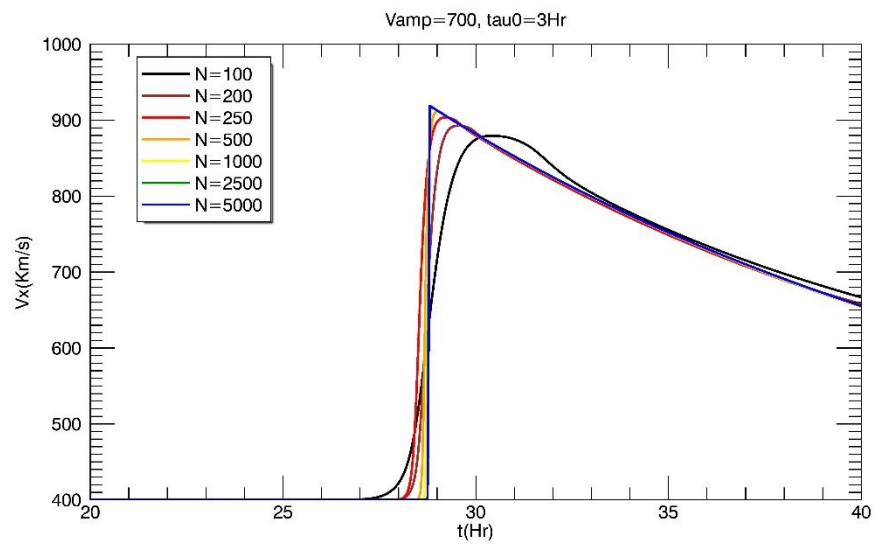
B=Bx

$$V_{AMP} = 700\ Km/s$$

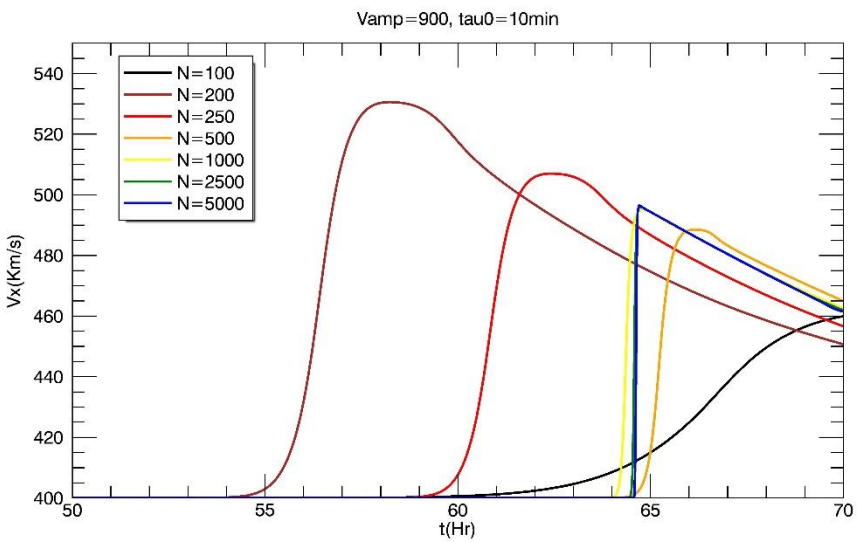


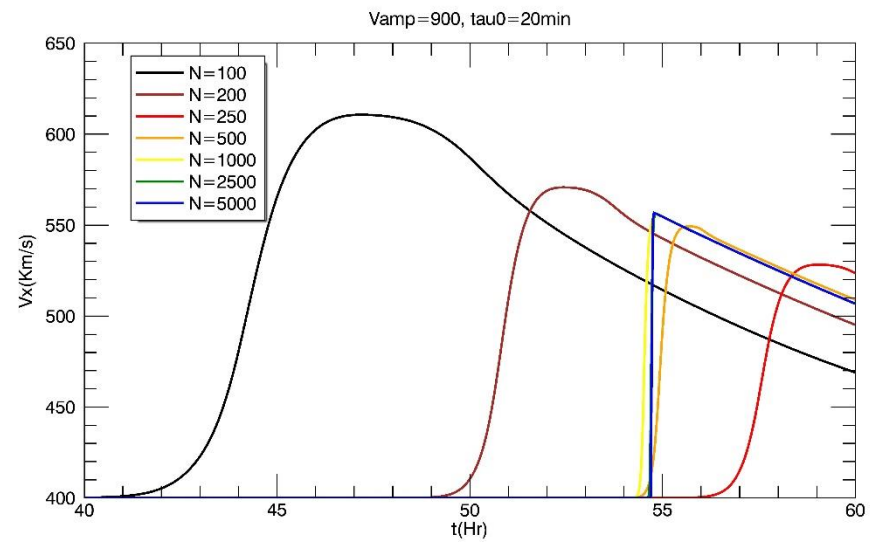


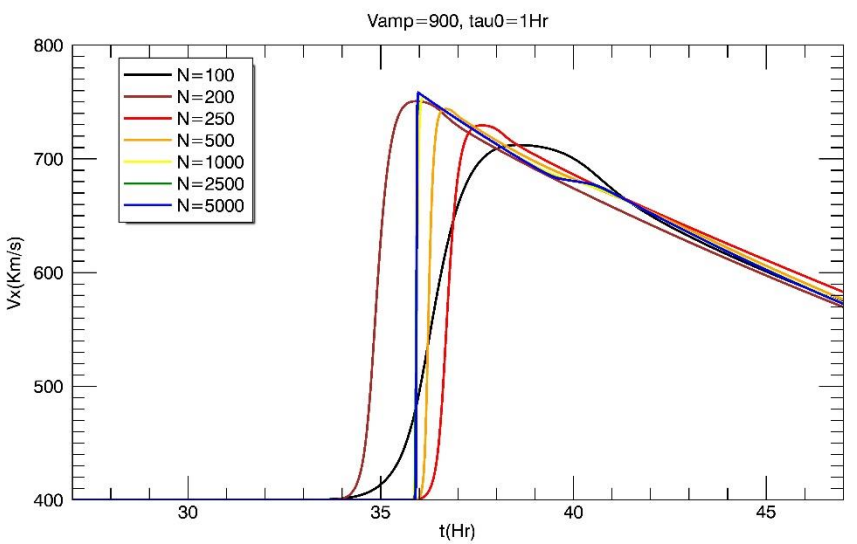


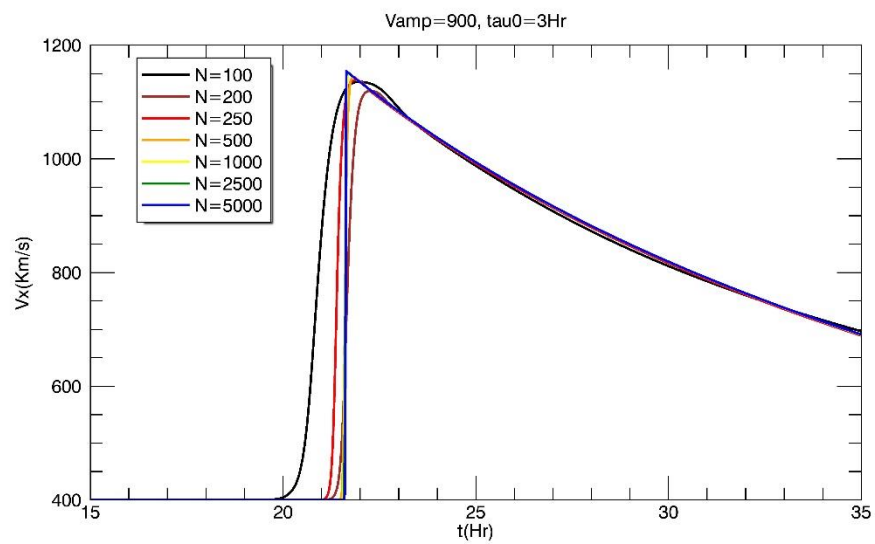


$V_{AMP} = 900\ Km/s$


Vamp=900, tau0=10min
N=100
N=200
N=250
N=500
N=1000
N=2500
N=5000
Vx(Km/s)
t(Hr)


Vamp=900, tau0=20min
N=100
N=200
N=250
N=500
N=1000
N=2500
N=5000
Vx(Km/s)
t(Hr)


Vamp=900, tau0=1Hr
N=100
N=200
N=250
N=500
N=1000
N=2500
N=5000
Vx(Km/s)
t(Hr)

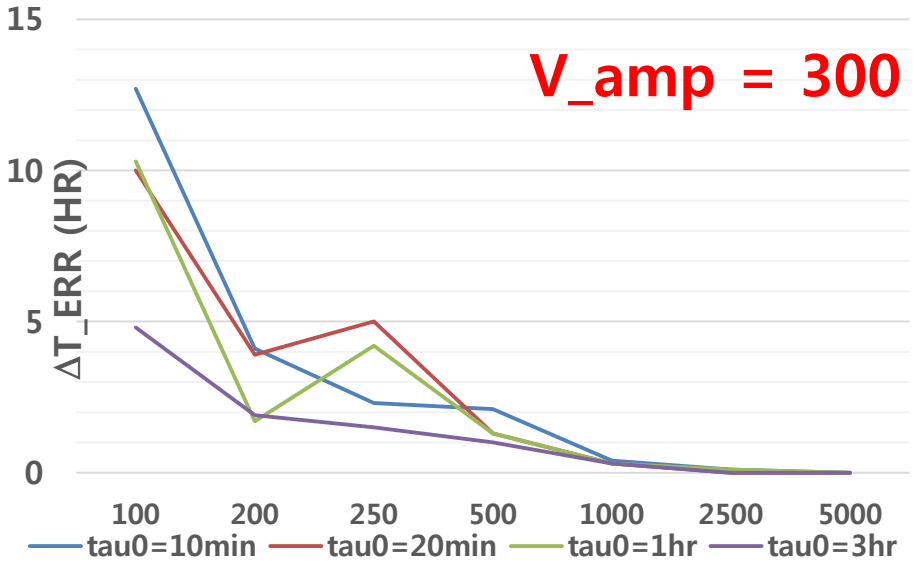

Vamp=900, tau0=3Hr
N=100
N=200
N=250
N=500
N=1000
N=2500
N=5000
Vx(Km/s)
t(Hr)

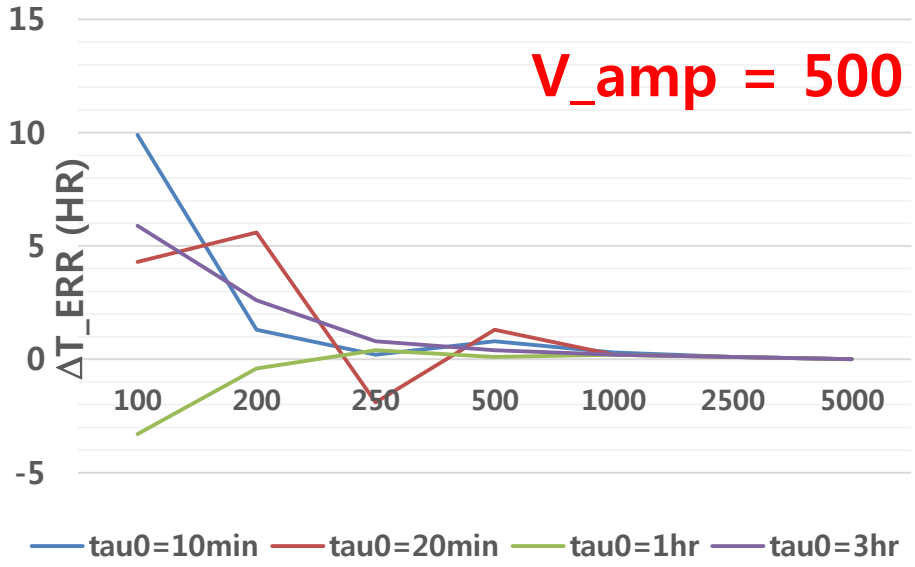
Arrival time at 0.99 AU (L1) – 1 % velocity

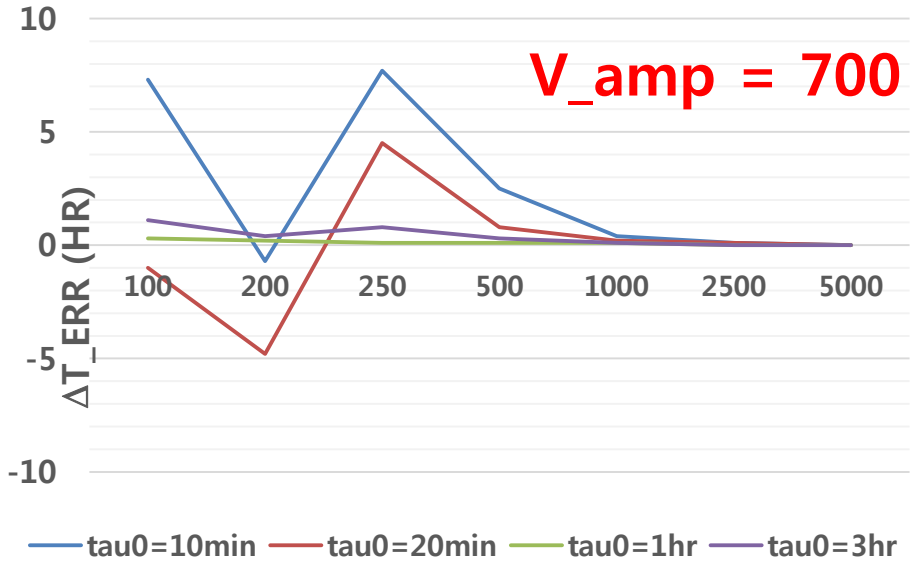
		N=100		N = 200		N = 250		N = 500		N = 1000		N = 2500		N=5000
		t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)
$V_{AMP} = 300$	$\tau_0 = 10\text{min}$	67.5	12.7	76.1	4.1	77.9	2.3	78.1	2.1	79.8	0.4	80.1	0.1	80.2
	$\tau_0 = 20min$	67.7	10	73.8	3.9	72.7	5	76.4	1.3	77.4	0.3	77.7	0	77.7
	$\tau_0 = 1\ Hr$	64.9	10.3	73.5	1.7	71	4.2	73.9	1.3	74.9	0.3	75.1	0.1	75.2
	$\tau_0 = 3\ Hr$	63.3	4.8	66.2	1.9	66.6	1.5	67.1	1	67.8	0.3	68.1	0	68.1
$V_{AMP} = 500$	$\tau_0 = 10\text{min}$	65.2	9.9	73.8	1.3	74.9	0.2	75.9	0.8	74.8	0.3	75	0.1	75.1
	$\tau_0 = 20min$	65.3	4.3	64	5.6	71.5	-1.9	68.3	1.3	69.4	0.2	69.5	0.1	69.6
	$\tau_0 = 1\ Hr$	62.3	-3.3	59.4	-0.4	58.6	0.4	58.9	0.1	58.8	0.2	58.9	0.1	59
	$\tau_0 = 3\ Hr$	49.3	5.9	52.6	2.6	54.4	0.8	54.8	0.4	55	0.2	55.1	0.1	55.2
$V_{AMP} = 700$	$\tau_0 = 10\text{min}$	62.5	7.3	70.5	-0.7	62.1	7.7	67.3	2.5	69.4	0.4	69.7	0.1	69.8
	$\tau_0 = 20min$	62.6	-1	66.4	-4.8	57.1	4.5	60.8	0.8	61.4	0.2	61.5	0.1	61.6
	$\tau_0 = 1\ Hr$	44.2	0.3	44.3	0.2	44.4	0.1	44.4	0.1	44.4	0.1	44.5	0	44.5
	$\tau_0 = 3\ Hr$	27.7	1.1	28.4	0.4	28	0.8	28.5	0.3	28.7	0.1	28.8	0	28.8

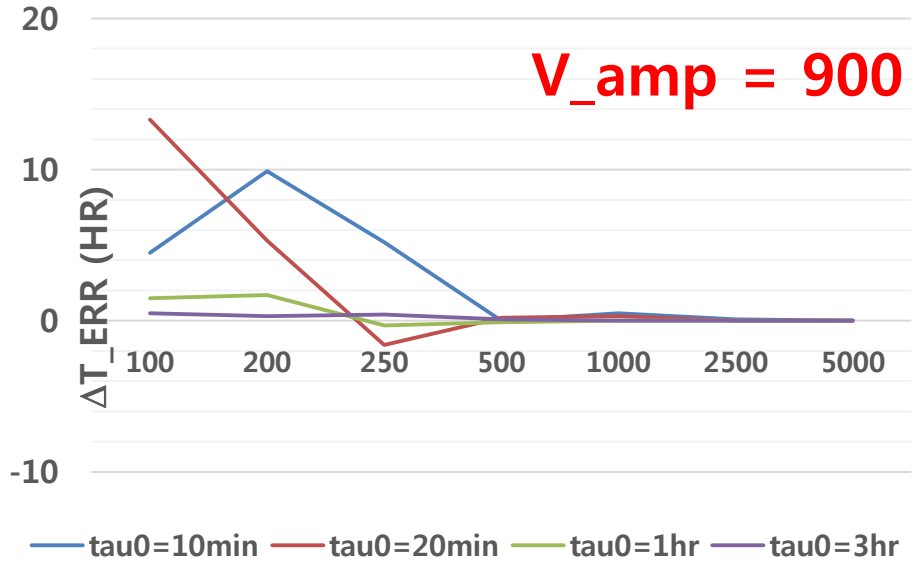
Arrival time at 0.99 AU (L1) – 1 % velocity (Cont'd)

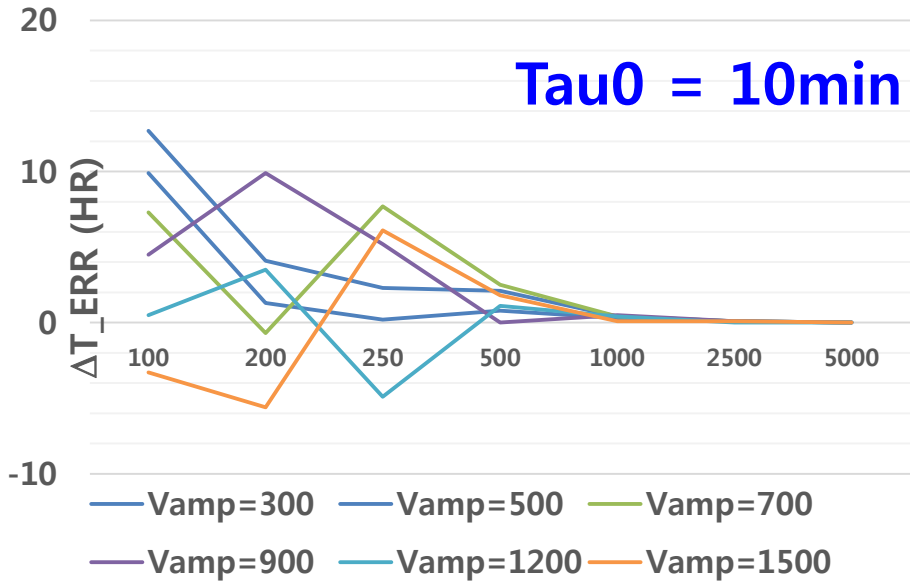
		N=100		N = 200		N = 250		N = 500		N = 1000		N = 2500		N=5000
		t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)	Δt_{err} (Hr)	t_{arr} (Hr)
$V_{AMP} = 900$	$\tau_0 = 10\text{min}$	60.1	4.5	54.7	9.9	59.4	5.2	64.6	0	64.1	0.5	64.5	0.1	64.6
	$\tau_0 = 20min$	41.4	13.3	49.4	5.3	56.3	-1.6	54.5	0.2	54.4	0.3	54.7	0	54.7
	$\tau_0 = 1\ Hr$	34.4	1.5	34.2	1.7	36.2	-0.3	36	-0.1	35.9	0	35.9	0	35.9
	$\tau_0 = 3\ Hr$	20.1	0.5	21.3	0.3	21.2	0.4	21.5	0.1	21.6	0	21.6	0	21.6
$V_{AMP} = 1200$	$\tau_0 = 10\text{min}$	56.8	0.5	53.8	3.5	62.2	-4.9	56.2	1.1	56.9	0.4	57.3	0	57.3
	$\tau_0 = 20min$	43	3.4	50.2	-3.8	46.1	0.3	45.1	1.3	46.2	0.2	46.4	0	46.4
	$\tau_0 = 1\ Hr$	28.7	-0.9	26.6	1.2	27.4	0.4	27.4	0.4	27.7	0.1	27.7	0.1	27.8
	$\tau_0 = 3\ Hr$	15.4	0.4	15.4	0.4	15.6	0.2	15.7	0.1	15.8	0	15.8	0	15.8
$V_{AMP} = 1500$	$\tau_0 = 10\text{min}$	54.3	-3.3	56.6	-5.6	44.9	6.1	49.2	1.8	50.9	0.1	50.9	0.1	51
	$\tau_0 = 20min$	49.8	-9.7	40.5	-0.4	39.1	1	40.4	0.3	40.2	0.1	40.1	0	40.1
	$\tau_0 = 1\ Hr$	22.5	0	21.5	1	21.6	0.9	22.3	0.2	22.4	0.1	22.5	0	22.5
	$\tau_0 = 3\ Hr$	11.7	0.5	12	0.2	12	0.2	12.1	0.1	12.1	0.1	12.1	0.1	12.2

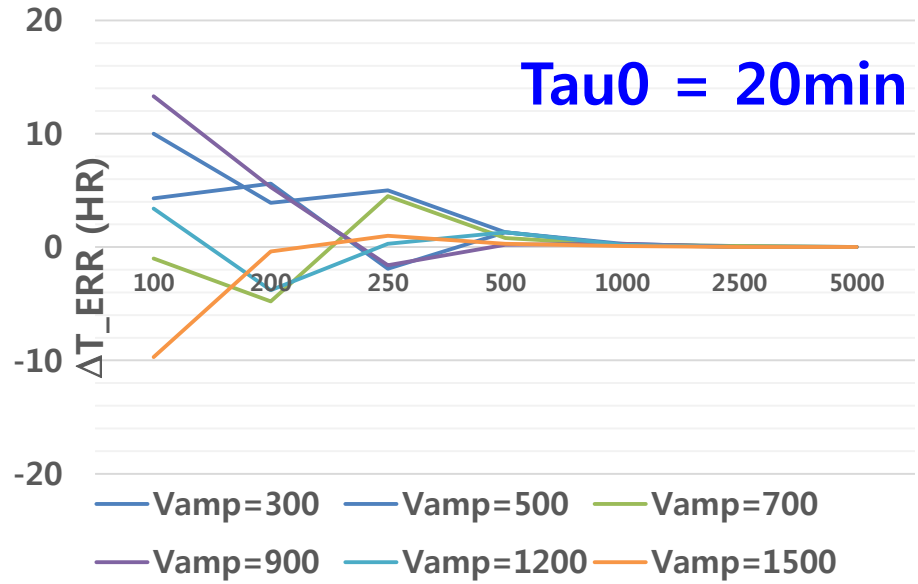

V_amp = 300
ΔT_ERR (HR)
15
10
5
0
100
200
250
500
1000
2500
5000
tau0=10min
tau0=20min
tau0=1hr
tau0=3hr

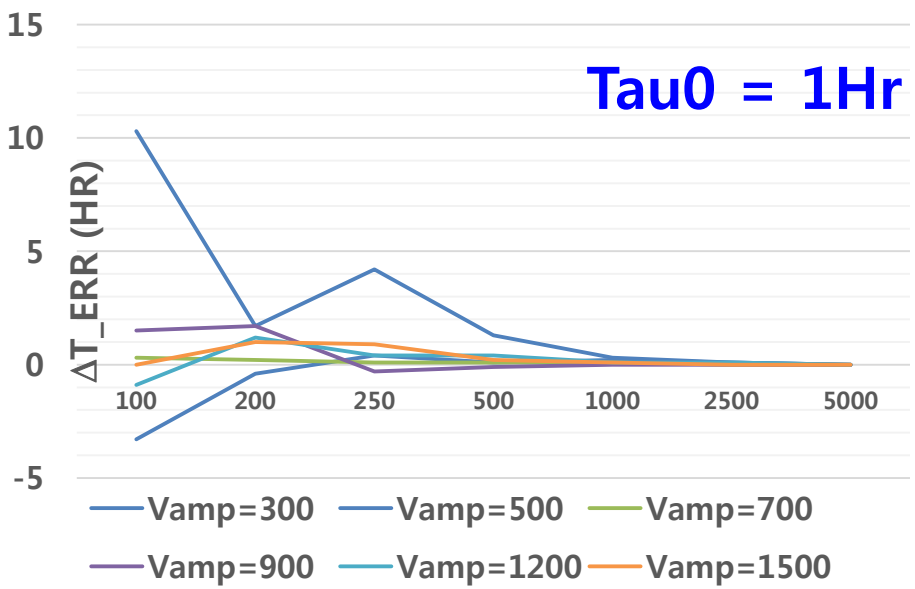

V_amp = 500
ΔT_ERR (HR)
15
10
5
0
-5
100
200
250
500
1000
2500
5000
tau0=10min
tau0=20min
tau0=1hr
tau0=3hr

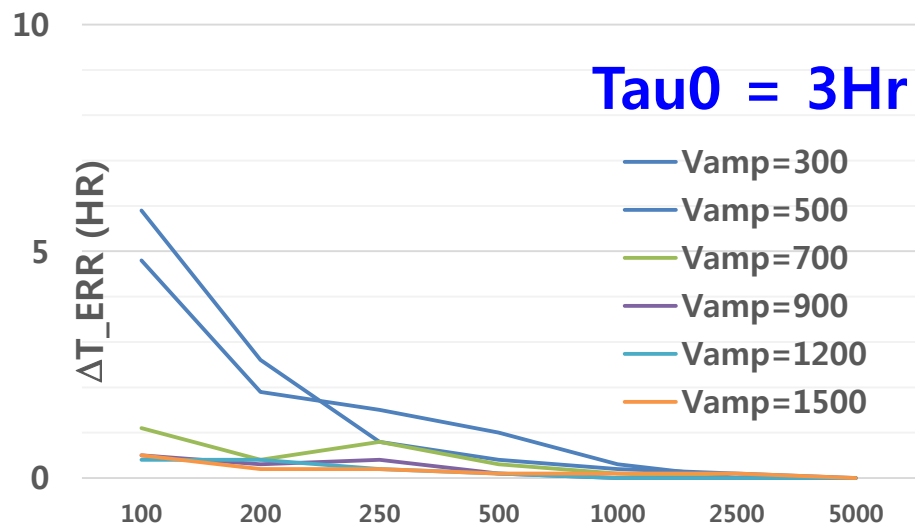

V_amp = 700
ΔT_ERR (HR)
10
5
0
-5
-10
100
200
250
500
1000
2500
5000
tau0=10min
tau0=20min
tau0=1hr
tau0=3hr


V_amp = 900
ΔT_ERR (HR)
20
10
0
-10
100
200
250
500
1000
2500
5000
tau0=10min
tau0=20min
tau0=1hr
tau0=3hr

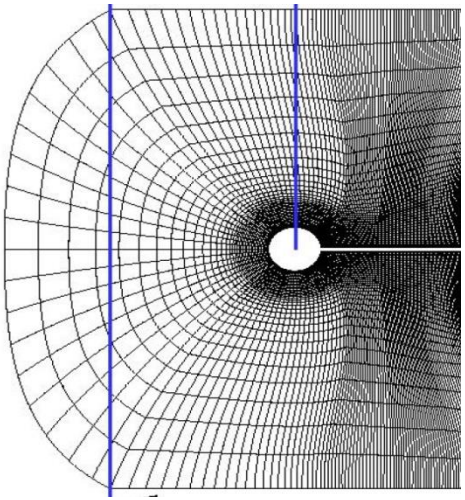

Tau0 = 10min
ΔT_ERR (HR)
20
10
0
-10
100
200
250
500
1000
2500
5000
Vamp=300
Vamp=500
Vamp=700
Vamp=900
Vamp=1200
Vamp=1500

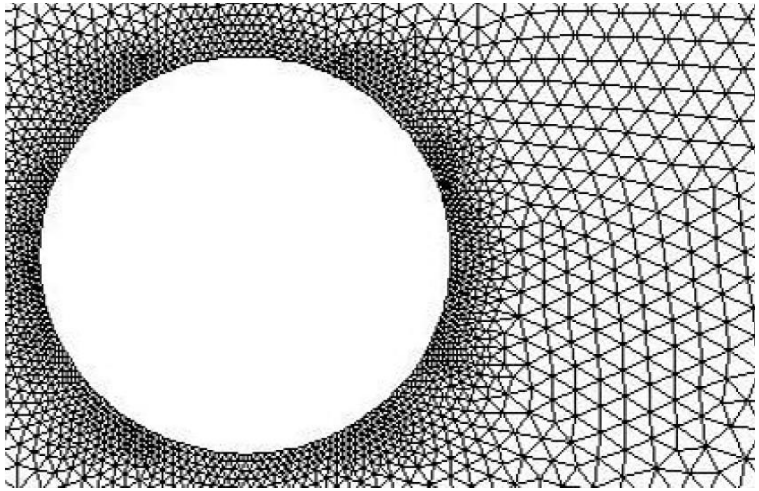

Tau0 = 20min
ΔT_ERR (HR)
20
10
0
-10
-20
100
200
250
500
1000
2500
5000
Vamp=300
Vamp=500
Vamp=700
Vamp=900
Vamp=1200
Vamp=1500

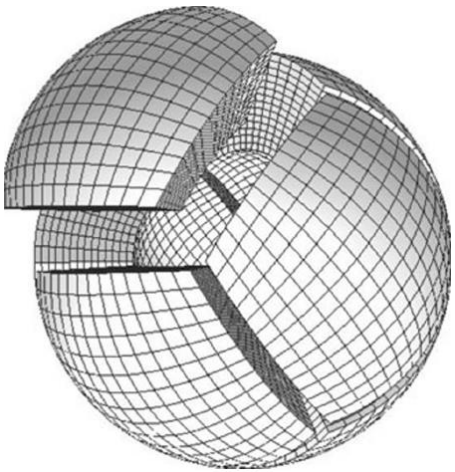

Tau0 = 1Hr
ΔT_ERR (HR)
15
10
5
0
-5
100
200
250
500
1000
2500
5000
Vamp=300
Vamp=500
Vamp=700
Vamp=900
Vamp=1200
Vamp=1500

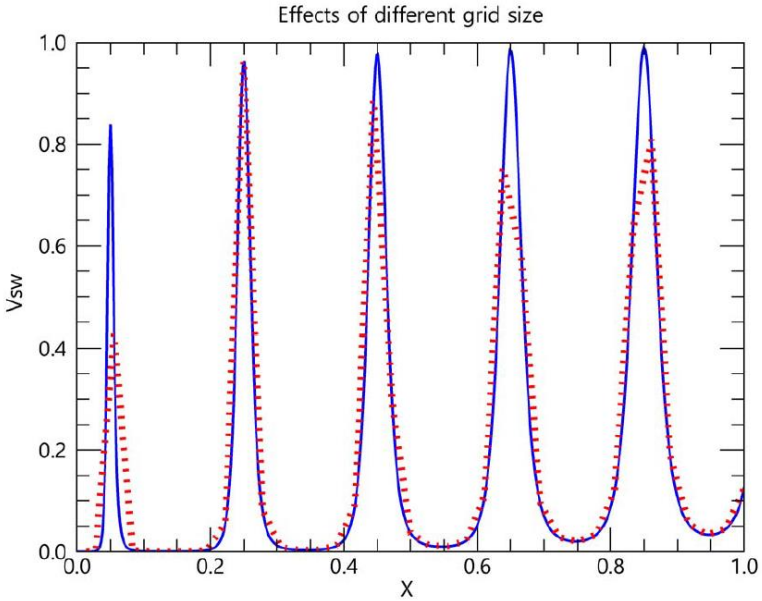

Tau0 = 3Hr
ΔT_ERR (HR)
10
5
0
100
200
250
500
1000
2500
5000
Vamp=300
Vamp=500
Vamp=700
Vamp=900
Vamp=1200
Vamp=1500

Some issues in numerical simulations





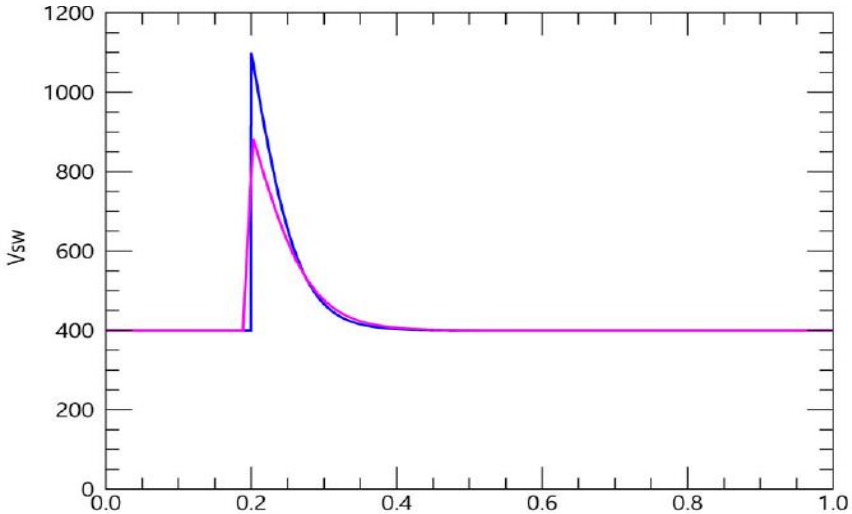


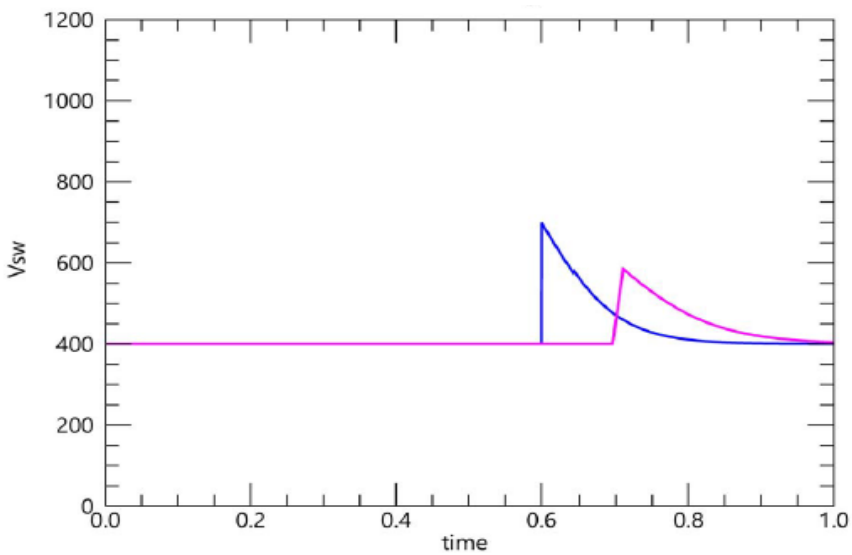


*** Spatial resolution (grid size) is associated with the structure (more grids for more details…).**

In dynamic system, time-dependent feature should be carefully reflected.

Some issues in numerical simulations

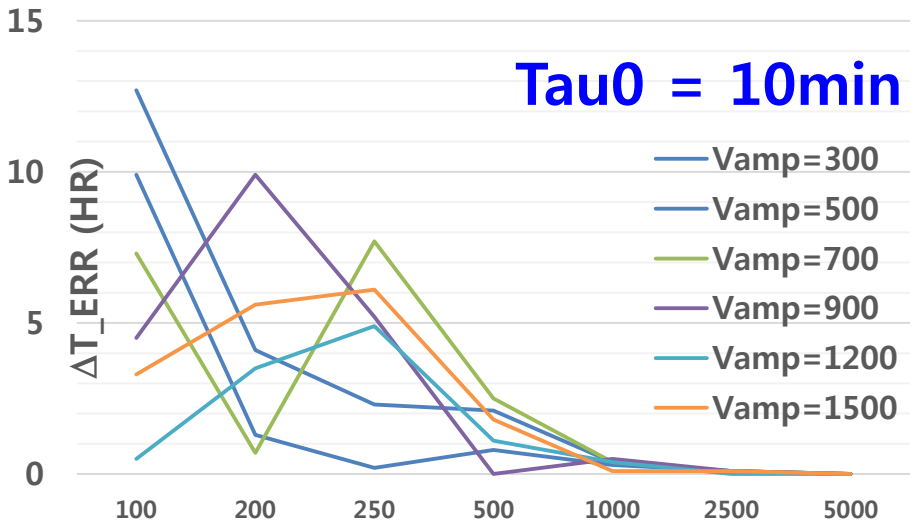


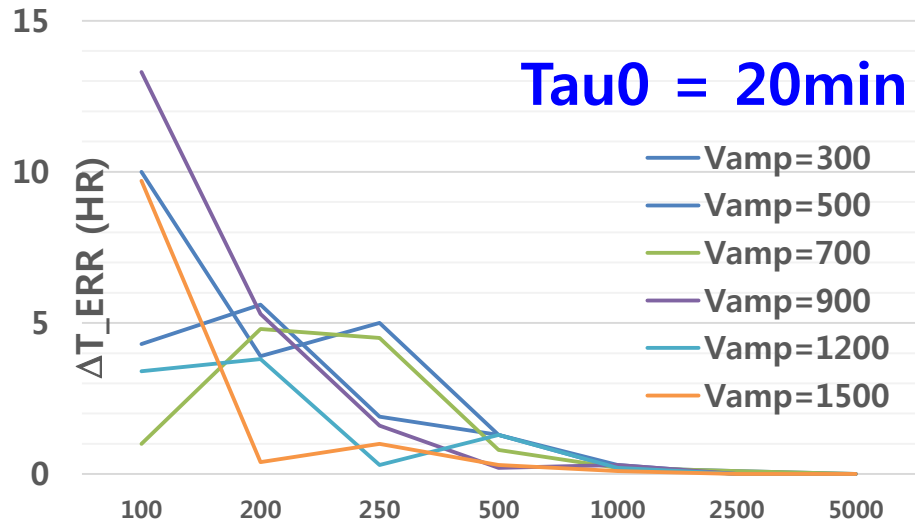


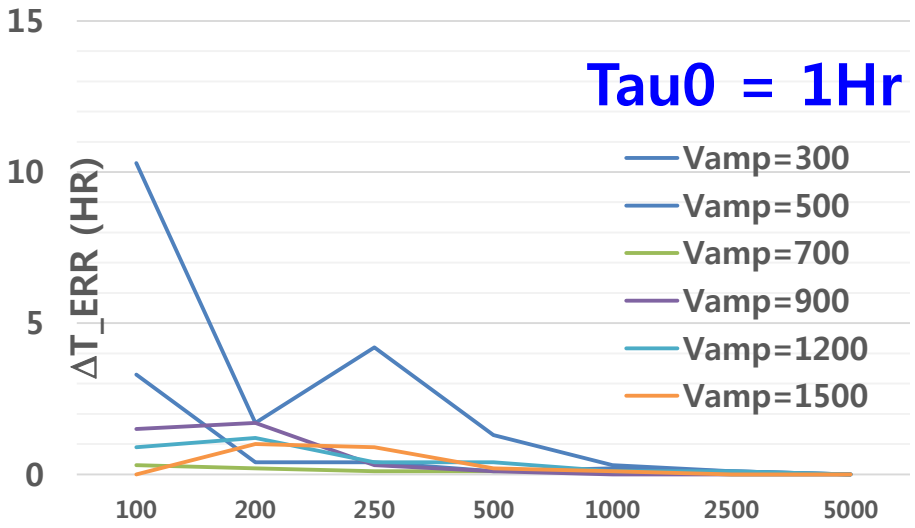
*** Numerical profiles become relatively sensitive to spacial resolution near the discontinuities or shocks.**

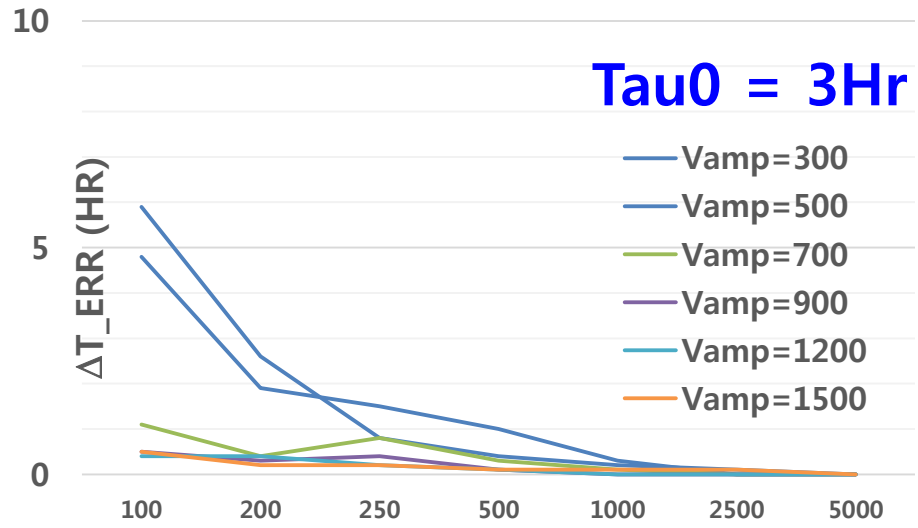
Near the leading edges, owing to
1) numerical diffusion and/or
2) differential speeds,
travel time differences to a distant location may become significant.

<Using Absolute Value>









Conclusion

- By comparing MHD simulations with an exact solution of MHD, we can check how much the simulations are accurate and precise.

- Numerical errors tend to be significant in the current MHD models from the Sun to the Earth such as CME if solar input is highly impulsive.

- Typical grid resolutions *(N=256~512/2AU)*, widely used in SW models, produce differences up to ~15 hours in arrival timing at the Earth.

- In order to maintain the steepening shock formation, the model needs far more grids than, e.g., N=1000 over the distance of 1 AU in average.

- As the source variations become more rapid, the numerical errors become larger and more dispersive.

- It is suggested that the model can be optimized in advance by considering each source characteristics such as rising time and amplitude.