Accuracy Issues in Space Weather MHD Models

* How to estimate computational errors?
* How to improve accuracy/precision?

Dong-Hun Lee, Kyung-Im Kim, Ensang Lee, Jinhy Hong
(Kyung Hee University, Korea)
Kyung Sun Park
(Chungbuk National Univ.)
Jaehun Kim, KiChang Yoon, Young Yun Kim
(Korean Space Weather Center)
- **Introduction:**
  * Effects of nonlinearity in a time-dependent system

- **Theory:**
  * What kinds of *exact* MHD solutions available?

- **Numerical tests:**
  * Theory *vs.* Numerical solutions
  * Effects of grid resolution

- **Conclusion**
- MHD numerical models are often studied in the interplanetary space. (*e.g.*, *ENLIL*)
- As realistic variations in the solar wind become often nonlinear, it is important to investigate time-dependent behaviors of the solar wind fluctuations.
Introduction

\[ f >> \delta f \]

- If the disturbance is \textit{linear} in a \textit{uniform} space, \( f_A = f_B \):

- If the disturbance is \textit{linear} in a \textit{nonuniform} space, only the effects of refraction/reflection are to change \( f_A \& f_B \):
- If the disturbance is **nonlinear**, $f_A$ and $f_B$ become differentiated even in a **uniform** space:

- The disturbance in the SW rest frame can evolve in a time-dependent manner, which is **far from the steady-state**.

  *ex) Rankine-Hugoniot relations
  Numerical test with stationary shock structure*
- For instance, the eq. of motion of MHD:

\[
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} v = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} (\vec{\nabla} \times \vec{B}) \times \vec{B}
\]

\[
\frac{\partial \vec{v}}{\partial t} \gg \vec{v} \cdot \vec{\nabla} v \quad : \text{Linear MHD waves} \quad f \gg \delta f
\]

\[
\frac{\partial \vec{v}}{\partial t} \ll \vec{v} \cdot \vec{\nabla} v \quad : \text{Steady-state } cf) \text{ R-H relations}
\]

\[
\frac{\partial \vec{v}}{\partial t} \sim \vec{v} \cdot \vec{\nabla} v \quad : \text{Nonlinear MHD waves} \quad f \sim \delta f
\]
Theory

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

\[
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + \frac{1}{\rho} \nabla p - \frac{1}{\rho} (\nabla \times \vec{B}) \times \vec{B} = 0
\]

\[
\frac{\partial p}{\partial t} + v \cdot \nabla p + \gamma p \nabla \cdot v = 0
\]

\[
\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) = 0
\]

The assumption of simple waves (similarity flow) is often used to obtain a solution.

Exact solution for the nonlinear MHD wave is available if it is a one-dimensional uniform system.
Theory

Sources => Simple waves => Shock waves

$(t > 0)$

$(t = \tau_s)$

* Simple waves $(0 < t \leq \tau_s)$

\[
\rho, \tilde{v}, \tilde{B}, p, \ldots = f_i(x - vt)
\]

or

\[
x - vt = g_i(\rho, \tilde{v}, \tilde{B}, p, \ldots)
\]

where $v = v(\rho, \tilde{v}, \tilde{B}, p, \ldots)$
* Simple waves

\( \text{if } \vec{k} \parallel \hat{x} \)

- Alfven waves:

\[ x - \left[ v_x + \frac{B_x}{\sqrt{\mu_0 \rho}} \right] t = \alpha (v_y, v_z) \]

where \( \rho, v_x, |B|, s = \text{const} \)

- Compressional waves:

\[ x - \left[ v_x + v_\pm \right] t = \beta (\rho) \]

where \( \frac{B_y}{B_z} = \frac{dv_y}{dv_z} = \text{const} \)

- Entropy waves:

\[ s = \gamma \left( x - v_x t \right) \]

where \( \rho, \vec{v}, \vec{B} = \text{const} \)

\( \alpha, \beta, \gamma \) are determined by IC and BC.
Theoretical solutions

- MHD equations

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} &= 0, \\
\frac{\partial B}{\partial t} + \frac{\partial (Bv)}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p^*}{\partial x} &= 0, \\
p^* &= p + \frac{B^2}{2\mu_0}
\end{align*}
\]

- Exact solution for the MHD wave  [Lee & Kim, JGR, 2000]

\[
\begin{align*}
x &= X(\tau) + (t - \tau) \left\{ \dot{X}(\tau) \pm C_T [\dot{X}(\tau)] \right\} \\
v &= \dot{X}(\tau) \\
v &= \pm 2V_A \sqrt{\frac{\rho}{\rho_0}} \left[ 1 + \frac{C^2}{V^2} \left( \frac{\rho}{\rho_0} \right)^{-\frac{1}{3}} \right]^\frac{3}{2} + v_0 \\
\frac{C_T}{V_A} &= \frac{v - v_0}{2V_A} - \frac{C^2}{2SV^2} \left( \frac{v - v_0}{V_A} \right)^{\frac{1}{3}}
\end{align*}
\]
Ex) piston-like motion

\[ X(t) = X_0 \tanh\left( \frac{t}{\tau_0} \right) \]
\[ v = \frac{X_0}{\tau_0} \sec h^2\left( \frac{t}{\tau_0} \right) \]
\[ \frac{x}{X_0} = \tanh \frac{\tau}{\tau_0} + \left( \frac{t}{\tau_0} - \frac{\tau}{\tau_0} \right) \sec h^2 \frac{\tau}{\tau_0} \pm \frac{C_T}{X_0 / \tau_0} \]
Numerical test vs. Exact solutions

Numerical solutions are corresponding to the exact solutions before the shock formation!
Numerical test vs. Exact solutions

Numerical solutions = Analytic solutions
Numerical test vs. Exact solutions
Numerical test vs. Exact solutions

\[<\text{nx}=2000>\]
\[V(\text{km/s})\]
\[X(\text{Re})\]

\[<\text{nx}=5000>\]
\[V(\text{km/s})\]
\[X(\text{Re})\]

\[<\text{nx}=10000>\]
\[V(\text{km/s})\]
\[X(\text{Re})\]

\[<\text{nx}=20000>\]
\[V(\text{km/s})\]
\[X(\text{Re})\]

20 / 40 / 60 / 80 / 100 (Hr)
Mean Absolute Difference of numerical error

\[
\langle |\delta F| \rangle = \frac{\sum_i |f(x_i) - F(x_i)|}{N}
\]
Introduction: **Numerical models**

### TABLE I
Basic Properties of Schemes

<table>
<thead>
<tr>
<th>Name</th>
<th>Base scheme</th>
<th>$\nabla \cdot B = 0$ constraint</th>
<th>Cost$^a$</th>
<th>Description</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-wave</td>
<td>Any</td>
<td>Truncation error</td>
<td>7%</td>
<td>Section 3</td>
<td>[31, 15]</td>
</tr>
<tr>
<td>Field-CT</td>
<td>Any</td>
<td>Conserves (15) and (27)</td>
<td>4%</td>
<td>Subsections 4.2 and 4.4</td>
<td>DW [11, 12]</td>
</tr>
<tr>
<td>Flux-CT</td>
<td>Any</td>
<td>Conserves (15) and (27)</td>
<td>5%</td>
<td>Subsections 4.3 and 4.4</td>
<td>BS [3]</td>
</tr>
<tr>
<td>Tr-flux-CT</td>
<td>One step TVD</td>
<td>Conserves (15) and (27)</td>
<td>6%</td>
<td>Subsections 4.3 and 4.4</td>
<td>RMJA [35]</td>
</tr>
<tr>
<td>Field-CD</td>
<td>Any</td>
<td>Conserves (26)</td>
<td>2%</td>
<td>Subsection 4.5</td>
<td>This paper</td>
</tr>
<tr>
<td>Flux-CD</td>
<td>Any</td>
<td>Conserves (26)</td>
<td>4%</td>
<td>Subsection 4.5</td>
<td>This paper</td>
</tr>
<tr>
<td>Projection</td>
<td>Any</td>
<td>Enforces (26) to be $&lt; \epsilon$</td>
<td>$\approx 20%$</td>
<td>Section 5</td>
<td>[7]</td>
</tr>
</tbody>
</table>

### TABLE II
Convergence of Average Errors for Alfvén Waves

<table>
<thead>
<tr>
<th>Travelling waves</th>
<th>Standing waves</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{32}$</td>
<td>$\delta_{32}$</td>
</tr>
<tr>
<td>$\delta_{34}$</td>
<td>$\delta_{34}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$\delta$</td>
</tr>
<tr>
<td>$\delta_{15}$</td>
<td>$\delta_{15}$</td>
</tr>
<tr>
<td>$\delta_{10}$</td>
<td>$\delta_{10}$</td>
</tr>
<tr>
<td>$\delta_{00}$</td>
<td>$\delta_{00}$</td>
</tr>
</tbody>
</table>

### TABLE IX
Numerical Errors Relative to the Most Accurate Scheme for Each Test

<table>
<thead>
<tr>
<th>Test Type</th>
<th>Project</th>
<th>Field-CD</th>
<th>Flux-CT</th>
<th>Flux-CD</th>
<th>8-wave</th>
<th>Tr-flux-CT</th>
<th>Field-CT</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotated 1D tests</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alfven travelling</td>
<td>1.000</td>
<td>2.177</td>
<td>2.491</td>
<td>2.491</td>
<td>1.005</td>
<td>1.290</td>
<td>3.562</td>
<td>1.002</td>
</tr>
<tr>
<td>Alfven standing</td>
<td>1.000</td>
<td>1.374</td>
<td>1.219</td>
<td>1.219</td>
<td>1.599</td>
<td>3.221</td>
<td>1.339</td>
<td>1.168</td>
</tr>
<tr>
<td>2D shock $\alpha = 63^\circ$</td>
<td>1.022</td>
<td>1.005</td>
<td>1.000</td>
<td>1.000</td>
<td>1.268</td>
<td>1.014</td>
<td>1.269</td>
<td>1.006</td>
</tr>
<tr>
<td>2D shock $\alpha = 45^\circ$</td>
<td>1.000</td>
<td>1.031</td>
<td>1.047</td>
<td>1.047</td>
<td>1.801</td>
<td>1.080</td>
<td>1.298</td>
<td>1.782</td>
</tr>
<tr>
<td>2.5D shock tube</td>
<td>1.000</td>
<td>1.137</td>
<td>1.234</td>
<td>1.234</td>
<td>1.023</td>
<td>1.132</td>
<td>1.392</td>
<td>1.333</td>
</tr>
<tr>
<td>True 2D tests</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Orsag $t = 1$</td>
<td>1.259</td>
<td>1.000</td>
<td>1.324</td>
<td>1.415</td>
<td>1.425</td>
<td>1.568</td>
<td>1.127</td>
<td>1.498</td>
</tr>
<tr>
<td>Orsag $t = 3.14$</td>
<td>1.132</td>
<td>1.000</td>
<td>1.188</td>
<td>1.233</td>
<td>1.411</td>
<td>1.383</td>
<td>1.187</td>
<td>1.893</td>
</tr>
<tr>
<td>Cloud-shock</td>
<td>1.007</td>
<td>1.069</td>
<td>1.000</td>
<td>1.036</td>
<td>1.013</td>
<td>1.072</td>
<td>1.348</td>
<td></td>
</tr>
<tr>
<td>Rotor $p = 1$</td>
<td>1.000</td>
<td>1.220</td>
<td>1.052</td>
<td>1.216</td>
<td>1.023</td>
<td>—</td>
<td>1.530</td>
<td>1.094</td>
</tr>
<tr>
<td>Rotor $p = 0.5$</td>
<td>1.000</td>
<td>1.058</td>
<td>1.098</td>
<td>1.116</td>
<td>1.050</td>
<td>1.230$^a$</td>
<td>1.289</td>
<td>1.071</td>
</tr>
<tr>
<td>Correction factor</td>
<td>$\approx 1.06$</td>
<td>1.006</td>
<td>1.018</td>
<td>1.013</td>
<td>1.023</td>
<td>1.022</td>
<td>1.014</td>
<td>1.000</td>
</tr>
</tbody>
</table>

$^a$ Obtained with the minmod limiter.
Simulation Box (Lx=1AU)

- Simulation parameters

\[
\begin{align*}
V_B &= 400 \text{ [km/s]} \\
C_{s0} &= 60 \text{ [km/s]} \\
V_{A0} &= 50 \text{ [km/s]} \\
\rho_0 &= 5 \text{ [m}_i \cdot \text{kg/cm}^3] \\
B_0 &= 5 \text{ [nT]} 
\end{align*}
\]

cf) N = 100, 200, 250, 500, 
1000, 2500, 5000, 10000

Sun

0.1 AU

Solar Wind (Vsw \sim 400 \text{ Km/s})

WSA-ENLIL input

Earth (or L1)
- Source motion

\[ v = v_0 \sin^2(w_0 t) \]

Where, \( w_0 = \frac{\pi}{\tau_0}, \ t = 0 \sim \tau_0 \)

\[ V_{Amp} \equiv \sqrt{\langle v^2 \rangle} = \frac{v_0}{\sqrt{2}} \]

\( \tau_o = 10 \text{ min}, \ 20 \text{ min}, \ 1 \text{ hr}, \ 3 \text{ hr}, \ldots \)

\[ V_{AMP} = 300, 500, 700, 900, 1200, 1500 \text{ km/s} \]

cf) \( N = 100, 200, 250, 500, 1000, 2500, 5000, 10000 \)
$V_{AMP} = 300 \text{ km/s}$ \quad $\tau_o = 10 \text{ min}$
\[ V_{\text{AMP}} = 300 \text{ Km/s} \]
$B = B_x$

$V_{AMP} = 300 \, Km/s$
$$V_{AMP} = 700 \text{ Km/s}$$
$B = B_x$

$V_{AMP} = 700$ Km/s
$V_{AMP} = 900 \text{ Km/s}$
<table>
<thead>
<tr>
<th>(V_{AMP} = 300)</th>
<th>(N=100)</th>
<th>(N = 200)</th>
<th>(N = 250)</th>
<th>(N = 500)</th>
<th>(N = 1000)</th>
<th>(N = 2500)</th>
<th>(N=5000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t_{arr}) (Hr)</td>
<td>(\Delta t_{err}) (Hr)</td>
<td>(t_{arr}) (Hr)</td>
<td>(\Delta t_{err}) (Hr)</td>
<td>(t_{arr}) (Hr)</td>
<td>(\Delta t_{err}) (Hr)</td>
<td>(t_{arr}) (Hr)</td>
<td>(\Delta t_{err}) (Hr)</td>
</tr>
<tr>
<td>(\tau_0 = 10) min</td>
<td>67.5</td>
<td>12.7</td>
<td>76.1</td>
<td>4.1</td>
<td>77.9</td>
<td>2.3</td>
<td>78.1</td>
</tr>
<tr>
<td>(\tau_0 = 20) min</td>
<td>67.7</td>
<td>10</td>
<td>73.8</td>
<td>3.9</td>
<td>72.7</td>
<td>5</td>
<td>76.4</td>
</tr>
<tr>
<td>(\tau_0 = 1) Hr</td>
<td>64.9</td>
<td>10.3</td>
<td>73.5</td>
<td>1.7</td>
<td>71</td>
<td>4.2</td>
<td>73.9</td>
</tr>
<tr>
<td>(\tau_0 = 3) Hr</td>
<td>63.3</td>
<td>4.8</td>
<td>66.2</td>
<td>1.9</td>
<td>66.6</td>
<td>1.5</td>
<td>67.1</td>
</tr>
</tbody>
</table>

| \(V_{AMP} = 500\) | \(\tau_0 = 10\) min | 65.2 | 9.9 | 73.8 | 1.3 | 74.9 | 0.2 | 75.9 | 0.8 |
| \(\tau_0 = 20\) min | 65.3 | 4.3 | 64 | 5.6 | 71.5 | -1.9 | 68.3 | 1.3 |
| \(\tau_0 = 1\) Hr | 62.3 | -3.3 | 59.4 | -0.4 | 58.6 | 0.4 | 58.9 | 0.1 |
| \(\tau_0 = 3\) Hr | 49.3 | 5.9 | 52.6 | 2.6 | 54.4 | 0.8 | 54.8 | 0.4 |

| \(V_{AMP} = 700\) | \(\tau_0 = 10\) min | 62.5 | 7.3 | 70.5 | -0.7 | 62.1 | 7.7 | 67.3 | 2.5 |
| \(\tau_0 = 20\) min | 62.6 | -1 | 66.4 | -4.8 | 57.1 | 4.5 | 60.8 | 0.8 |
| \(\tau_0 = 1\) Hr | 44.2 | 0.3 | 44.3 | 0.2 | 44.4 | 0.1 | 44.4 | 0.1 |
| \(\tau_0 = 3\) Hr | 27.7 | 1.1 | 28.4 | 0.4 | 28 | 0.8 | 28.5 | 0.3 |

Arrival time at 0.99 AU (L1) – 1 % velocity
## Arrival time at 0.99 AU (L1) – 1 % velocity (Cont’d)

<table>
<thead>
<tr>
<th>$V_{AMP}$</th>
<th>$\tau_0$ (min)</th>
<th>$N=100$</th>
<th>$N=200$</th>
<th>$N=250$</th>
<th>$N=500$</th>
<th>$N=1000$</th>
<th>$N=2500$</th>
<th>$N=5000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>900</td>
<td>10</td>
<td>60.1</td>
<td>54.7</td>
<td>59.4</td>
<td>64.6</td>
<td>64.1</td>
<td>64.5</td>
<td>64.6</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>41.4</td>
<td>49.4</td>
<td>56.3</td>
<td>54.5</td>
<td>54.4</td>
<td>54.7</td>
<td>54.7</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>34.4</td>
<td>34.2</td>
<td>36.2</td>
<td>36.0</td>
<td>35.9</td>
<td>35.9</td>
<td>35.9</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>20.1</td>
<td>21.3</td>
<td>21.2</td>
<td>21.5</td>
<td>21.6</td>
<td>21.6</td>
<td>21.6</td>
</tr>
<tr>
<td>1200</td>
<td>10</td>
<td>56.8</td>
<td>53.8</td>
<td>62.2</td>
<td>56.2</td>
<td>56.9</td>
<td>57.3</td>
<td>57.3</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>43.0</td>
<td>50.2</td>
<td>46.1</td>
<td>45.1</td>
<td>46.2</td>
<td>46.4</td>
<td>46.4</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>28.7</td>
<td>26.6</td>
<td>27.4</td>
<td>27.4</td>
<td>27.7</td>
<td>27.7</td>
<td>27.8</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>15.4</td>
<td>15.4</td>
<td>15.6</td>
<td>15.7</td>
<td>15.8</td>
<td>15.8</td>
<td>15.8</td>
</tr>
<tr>
<td>1500</td>
<td>10</td>
<td>54.3</td>
<td>56.6</td>
<td>44.9</td>
<td>49.2</td>
<td>50.9</td>
<td>50.9</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>49.8</td>
<td>40.5</td>
<td>39.1</td>
<td>40.4</td>
<td>40.2</td>
<td>40.1</td>
<td>40.1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>22.5</td>
<td>21.5</td>
<td>21.6</td>
<td>22.3</td>
<td>22.4</td>
<td>22.5</td>
<td>22.5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>11.7</td>
<td>12</td>
<td>12</td>
<td>12.1</td>
<td>12.1</td>
<td>12.1</td>
<td>12.2</td>
</tr>
</tbody>
</table>
Graphs showing $\Delta T_{ERR}$ for different V_amplitude ($V_{amp}$) values with varying tau0 (time constants). The graphs illustrate the error in temperature over time for different values of $V_{amp}$ ranging from 300 to 900 with tau0 values of 10min, 20min, 1hr, and 3hr.
Some issues in numerical simulations

* Spatial resolution (grid size) is associated with the structure (more grids for more details...).

In dynamic system, time-dependent feature should be carefully reflected.
Some issues in numerical simulations

* Numerical profiles become relatively sensitive to spacial resolution near the discontinuities or shocks.

Near the leading edges, owing to
1) numerical diffusion and/or
2) differential speeds,
travel time differences to a distant location may become significant.
<Using Absolute Value>

 Tau0 = 10min

 [Graph showing various lines representing different Vamp values for different time points]

 Tau0 = 20min

 [Graph showing various lines representing different Vamp values for different time points]

 Tau0 = 1Hr

 [Graph showing various lines representing different Vamp values for different time points]

 Tau0 = 3Hr

 [Graph showing various lines representing different Vamp values for different time points]
Conclusion

- By comparing MHD simulations with an exact solution of MHD, we can check how much the simulations are accurate and precise.

- Numerical errors tend to be significant in the current MHD models from the Sun to the Earth such as CME if solar input is highly impulsive.

- Typical grid resolutions \( (N=256\sim 512/2AU) \), widely used in SW models, produce differences up to \( \sim 15 \) hours in arrival timing at the Earth.

- In order to maintain the steepening shock formation, the model needs far more grids than, e.g., \( N=1000 \) over the distance of 1 AU in average.

- As the source variations become more rapid, the numerical errors become larger and more dispersive.

- It is suggested that the model can be optimized in advance by considering each source characteristics such as rising time and amplitude.